HOMOGENEOUS IRREDUCIBLE SUPERMANIFOLDS AND GRADED LIE SUPERALGEBRAS

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Abstract. A depth one grading $\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^\ell$ of a finite dimensional Lie superalgebra $\mathfrak{g}$ is called nonlinear irreducible if the isotropy representation $\mathrm{ad}_{\mathfrak{g}^0}|_{\mathfrak{g}^{-1}}$ is irreducible and $\mathfrak{g}^1 \neq (0)$. An example is the full prolongation of an irreducible linear Lie superalgebra $\mathfrak{g}^0 \subset \mathfrak{gl}(\mathfrak{g}^{-1})$ of finite type with non-trivial first prolongation. We prove that a complex Lie superalgebra $\mathfrak{g}$ which admits a depth one transitive nonlinear irreducible grading is a semisimple Lie superalgebra with the socle $\mathfrak{s} \otimes \Lambda(\mathbb{C}^n)$, where $\mathfrak{s}$ is a simple Lie superalgebra, and we describe such gradings. The graded Lie superalgebra $\mathfrak{g}$ defines an isotropy irreducible homogeneous supermanifold $M = G/G_0$ where G, G_0 are Lie supergroups respectively associated with the Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}_0 := \bigoplus_{p \geq 0} \mathfrak{g}^p$.

1. Introduction

An homogeneous manifold $M = G/H$ of a connected Lie group G with a connected stability subgroup H is called (isotropy) *nonlinear* if the isotropy representation $j : H \to GL(T_oM) \simeq GL(\mathfrak{g}/\mathfrak{h})$ has a positive dimensional kernel or, in other words, there is no system of local coordinates near $o = eH \in M$ which linearizes the action of H. A nonlinear homogeneous manifold $M = G/H$ is called *irreducible* (resp. *primitive*) if the isotropy group $j(H)$ is irreducible (resp. the stabilizer H is a maximal connected subgroup of G).

All nonlinear irreducible homogeneous manifolds had been classified by S. Kobayashi and T. Nagano (see [18, 19]) and the nonlinear primitive homogeneous manifolds by T. Ochiai (see [21]). It turns out that any complex nonlinear primitive manifold $M = G/H$ is the quotient of a complex simple Lie group G by a maximal parabolic subgroup $H = P_\alpha$ associated to a simple root α (the semisimple part of P_α is the semisimple regular subgroup of G associated with the Dynkin diagram of G with deleted root α). Any real nonlinear primitive homogeneous manifold has the form $M^\sigma = G^\sigma/P^\sigma_\alpha$ where G^σ is a real form of G defined by an anti-linear involution of the Lie algebra $\mathfrak{g} = Lie(G)$ which preserves the subalgebra $\mathfrak{p}_\alpha = Lie(P_\alpha)$ and P^σ_α is the parabolic subgroup associated with the fixed point set $\mathfrak{p}^\sigma_\alpha$ of σ in $\mathfrak{p}_\alpha$. The nonlinear primitive manifold $M = G/P_\alpha$ or its real form $M^\sigma = G^\sigma/P^\sigma_\alpha$

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is irreducible if and only if the Dynkin mark $m_\alpha = 1$. We recall that the Dynkin marks are the coordinates of the maximal root with respect to the basis of simple roots.

The problem of classifying the nonlinear primitive manifolds reduces to the classification of the nonlinear primitive transitive Lie algebras (L, L_0). Here L is a Lie algebra, L_0 a maximal subalgebra of L which does not contain any nontrivial ideal of L (this condition is called effectivity or transitivity) and such that the isotropy representation $\mathrm{ad}_{L_0}|_{L/L_0}$ has a kernel $L_1 \neq (0)$. Moreover, it corresponds to an irreducible homogeneous manifold if the isotropy representation $\mathrm{ad}_{L_0}|_{L/L_0}$ is irreducible. The subalgebra L_0 defines a natural filtration of L (see [32]) and the study of the associated $\mathbb{Z}$-graded Lie algebra is a crucial step for the classification of the primitive homogeneous manifolds.

In the present paper, we deal with the similar problem for supermanifolds. Like for the homogeneous manifolds, a homogeneous supermanifold can be described as a quotient $M = G/H$ of a Lie supergroup G by a Lie supersubgroup H and the pair (G, H) is essentially determined by the associated pair $(L = Lie(G), L_0 = Lie(H))$ of Lie superalgebras, see e.g. [26] for more details. The condition that $M = G/H$ is nonlinear primitive or irreducible can also be also rephrased in terms of the pair (L, L_0) of Lie superalgebras, see §2.1. We prove in §2 that if (L, L_0) is a complex nonlinear primitive Lie superalgebra, then the Lie superalgebra L is semisimple and its socle $\mathrm{soc}(L)$ is necessarily of the form $\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$, where $\mathfrak{s}$ is a simple Lie superalgebra and $\Lambda = \Lambda(V^*) = \mathbb{C}1 \oplus \Lambda^+$ is the algebra of exterior forms on $V = \mathbb{C}^n$ for some $n \geq 0$ (we recall the the socle of a Lie superalgebra L is the sum of all minimal ideals of L). According to V. G. Kac in [14] (see also [9]), any such Lie superalgebra L is a subalgebra of the Lie superalgebra of derivations of the socle

$$\mathrm{der}(\mathfrak{s}^\Lambda) = \mathrm{der}(\mathfrak{s}) \otimes \Lambda \ni 1 \otimes W ,$$

where $W = \mathrm{der}(\Lambda) = \Lambda \otimes \partial_V$ is the algebra of derivations of Λ and ∂_V is the space of "constant odd supervector fields". Moreover, the condition of semisimplicity for L is equivalent to the fact that the restriction to L of the natural projection

$$\mathrm{der}(\mathfrak{s}^\Lambda) \longrightarrow \mathrm{der}(\mathfrak{s}^\Lambda)/(\mathrm{der}(\mathfrak{s}) \otimes \Lambda + 1 \otimes \Lambda^+ \partial_V) \simeq \partial_V$$

is surjective. In this paper any such semisimple Lie superalgebra $L \subset \mathrm{der}(\mathfrak{s}^\Lambda)$ is called an *intermediate admissible* Lie superalgebra.

Any maximal subalgebra L_0 of L defines a filtration (see [32, 4])

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$$

and the associated $\mathbb{Z}$-graded transitive Lie superalgebra of depth $d(\mathfrak{g}) = d$:

$$\mathfrak{g} = \mathrm{gr}(L) = L_{-d}/L_{-d+1} \oplus \cdots \oplus L_0/L_1 \oplus L_1/L_2 \oplus \cdots \ .$$

The problem of classifying the nonlinear primitive Lie superalgebras (L, L_0) where L is a semisimple Lie superalgebra as above is an involved problem.

The first step for its solution is to describe transitive primitive or irreducible $\mathbb{Z}$-graded Lie superalgebras. In the case of Lie algebras, any finite dimensional primitive filtered Lie algebra L is isomorphic to the associated $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \mathrm{gr}(L)$ (see [20, 21]). A similar statement is also true for the infinite-dimensional primitive Lie algebras (see [30, 13, 23]).

For Lie superalgebras, the situation is more complicated and a filtered Lie superalgebra can often be obtained by an appropriate deformation of the bracket of the associated $\mathbb{Z}$-graded Lie superalgebra (see [7]; see also [10] for an instance of this phenomenon in the context of supersymmetric field theories). The full classification of the infinite-dimensional primitive (filtered) Lie superalgebras was obtained by N. Cantarini, S.-J. Cheng and V. Kac in [4, 8].

The present paper gives a full description of the *finite dimensional nonlinear transitive irreducible $\mathbb{Z}$-graded* Lie superalgebras $\mathfrak{g} = \bigoplus_{i=-1}^{\ell} \mathfrak{g}^i$. In this case, the grading of $\mathfrak{g}$ defines a filtration of the form $\mathfrak{g}_{-1} = \mathfrak{g} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots$ where $\mathfrak{g}_p := \bigoplus_{i\geq p} \mathfrak{g}^i$ and the associated pair $(L, L_0) = (\mathfrak{g}, \mathfrak{g}_0)$ is an irreducible Lie superalgebra which is nonlinear, that is with $L_1 = \mathfrak{g}_1 = \bigoplus_{i\geq 1} \mathfrak{g}^i \neq (0)$. The study of the filtered deformations of $\mathfrak{g} = \bigoplus_{i\geq -1} \mathfrak{g}^i$ will be the content of a future work.

We remark that the *full* prolongation of a linear Lie superalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ which acts irreducibly on a supervector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ can be defined as the *maximal* depth one $\mathbb{Z}$-graded transitive Lie superalgebra $V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots$ which extends $V \oplus \mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$. All such full prolongations which are infinite-dimensional were described in [17, 8]. On the other hand the full prolongation of an irreducible linear Lie superalgebra $\mathfrak{g}$ with nontrivial first prolongation $\mathfrak{g}^{(1)} \neq (0)$ is finite dimensional if and only if there are no elements in $\mathrm{ad}_{\mathfrak{g}_0}|_{(\mathfrak{g}_{-1})_{\bar{0}}}$ of rank one, see [1, 31], and the class of all such graded Lie superalgebras is just a subclass of the finite dimensional nonlinear transitive irreducible graded Lie superalgebras described below. This gives, in particular, a description of all the irreducible representations of Lie superalgebras with a nontrivial first prolongation. This description is clearly of interest *per se* but we would also like to mention that some representations with nontrivial first prolongation had recently played an important rôle in a partial generalization to the case of supermanifolds of the classical result of M. Berger on the possible irreducible holonomy algebras of Riemannian manifolds (see [11, 12]).

The main results of this paper are Theorem 3.14 in §3 and Theorem 4.2 in §4 and the description in §3.5.2, §3.5.3 and §3.5.4 of all the $\mathbb{Z}$-gradings of depth one of the Lie superalgebras $\mathrm{der}(\mathfrak{s})$ of derivations of simple Lie superalgebra $\mathfrak{s}$. We summarize the main results in the following Theorem 1.1 for the reader's convenience. To state it, we recall that the Lie superalgebra $\mathrm{der}(\mathfrak{s})$ always admits a canonical decomposition into the semidirect sum $\mathrm{der}(\mathfrak{s}) = \mathfrak{s} \ni \mathrm{out}(\mathfrak{s})$ of the ideal of inner derivations (identified with $\mathfrak{s}$) and a complementary subalgebra $\mathrm{out}(\mathfrak{s})$ of outer derivations (see [14]).

Theorem 1.1. *Let $\mathfrak{g} = \bigoplus_{i=-1}^{\ell} \mathfrak{g}^i$ be a transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebra of depth one. Then $\mathfrak{g}$ is a semisimple Lie superalgebra with socle $\mathrm{soc}(\mathfrak{g}) = \mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$ where $\mathfrak{s}$ is a uniquely determined simple Lie superalgebra and Λ is the algebra of exterior forms on $V = \mathbb{C}^n$, for $n \geq 0$. The superalgebra $\mathfrak{g}$ is a graded subalgebra of the Lie superalgebra*

$$\begin{aligned}\mathrm{der}(\mathfrak{s}^\Lambda) &= \mathrm{der}(\mathfrak{s}) \otimes \Lambda \ni 1 \otimes W \\ &= \mathfrak{s}^\Lambda \ni \mathrm{out}(\mathfrak{s}^\Lambda)\ ,\end{aligned}$$

where $W = \mathrm{der}(\Lambda)$ and $\mathrm{out}(\mathfrak{s}^\Lambda) = \mathrm{out}(\mathfrak{s}) \otimes \Lambda \ni 1 \otimes W$, with one of the following gradings:

- (I) *$\mathfrak{s} = \bigoplus \mathfrak{s}^p$ has a $\mathbb{Z}$-grading of depth 1 and $\Lambda = \Lambda^0$ and $W = W^0$ have the trivial gradings;*
- (II) *$\mathfrak{s} = \mathfrak{s}^0$ has the trivial grading whereas Λ and W have the gradings defined by a grading $V = V^0 \oplus V^1$ with $\dim V^1 = 1$.*

If $n = 0$ then the socle of $\mathfrak{g}$ is a simple Lie superalgebra $\mathfrak{s}$, the grading of $\mathfrak{g}$ is as in (I) and $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie superalgebra of the form $\mathfrak{g} = \mathfrak{s} \ni F$ where $F \subset \mathrm{out}(\mathfrak{s})$ is a nonnegatively $\mathbb{Z}$-graded subalgebra of $\mathrm{out}(\mathfrak{s})$. The relevant $\mathbb{Z}$-gradings of $\mathrm{out}(\mathfrak{s})$ and $\mathrm{der}(\mathfrak{s})$ are described in §3.5.2, §3.5.3 and §3.5.4. If $n > 0$ then $\mathfrak{g}$ is the semidirect sum $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$, where $F = \bigoplus F^p$ is a nonnegatively graded subalgebra of the Lie superalgebra $\mathrm{out}(s^\Lambda)$ which is admissible, that is the natural projection from F to ∂_V is surjective.

Conversely any grading as in (I) or (II) defines a transitive nonlinear and irreducible grading of depth one of the Lie superalgebra $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ for any nonnegatively graded subalgebra F of $\mathrm{out}(\mathfrak{s}^\Lambda)$ which is admissible.

The paper is organized as follows. In §2 we give the main preliminary definitions and results on nonlinear primitive Lie superalgebras and recall the description of the semisimple Lie superalgebras. In §3.1, §3.2, §3.3 and §3.4 we give some basic results on irreducible $\mathbb{Z}$-graded Lie superalgebras, describe the gradings of the Grassmann algebra Λ and its Lie superalgebra $W = \mathrm{der}(\Lambda)$ of derivations and, finally, give the description of the $\mathbb{Z}$-gradings of the semisimple Lie superalgebras with socle $\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$. Section 3.5 contains Theorem 3.14 and the description of the gradings of $\mathrm{der}(\mathfrak{s})$ associated with the depth one gradings of the simple Lie superalgebras $\mathfrak{s}$. We mention that, besides the well-known last grading of the strange Lie superalgebra $\mathfrak{s} = \mathfrak{spe}(n)$ described in Table 7, there is another interesting grading that is of depth $d = 1$ and height $\ell = 2$: it is a grading on a subalgebra of the algebra $\mathrm{der}(\mathfrak{s})$ of derivations of the basic Lie superalgebra $\mathfrak{s} = \mathfrak{psl}(2|2)$ (see Table 5). We apply these results in Section 4 to the description in Theorem 4.2 of the depth one transitive nonlinear irreducible gradings of semisimple Lie superalgebras and conclude with some observations.

Notations.

Given any supervector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by

$$\Pi V = (\Pi V)_{\bar{0}} \oplus (\Pi V)_{\bar{1}}$$

the supervector space with opposite parity, that is

$$(\Pi V)_{\bar{0}} = V_{\bar{1}} \ , \qquad (\Pi V)_{\bar{1}} = V_{\bar{0}} \tag{1.1}$$

as (non-super) vector spaces. We set $\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \Pi(\mathbb{C}^n)$.

The tensor product $V \otimes V'$ of two supervector spaces has a natural structure of supervector space given by

$$(V \otimes V')_{\bar{0}} = (V_{\bar{0}} \otimes V'_{\bar{0}}) \oplus (V_{\bar{1}} \otimes V'_{\bar{1}})\,,\ (V \otimes V')_{\bar{1}} = (V_{\bar{0}} \otimes V'_{\bar{1}}) \oplus (V_{\bar{1}} \otimes V'_{\bar{0}})\ .$$

In §3.5.2-§3.5.3 (and only there) we use the symbols $\Lambda(\mathbb{C}^{m|n})$ and $S(\mathbb{C}^{m|n})$ to denote the exterior and symmetric products of $V = \mathbb{C}^{m|n}$ in the super sense.

We denote finite-dimensional simple Lie superalgebras according to the conventions in e.g. [28, 6] (some authors use different conventions, especially for classical Lie superalgebras; cf. [14, 27]).

The projectivization of a linear Lie superalgebra $\mathfrak{g}$ which contains the scalar matrices is denoted by $\mathfrak{pg} = \mathfrak{g}/\mathbb{C}\,\mathrm{Id}$, if it does not contain $\mathbb{C}\,\mathrm{Id}$ we set $\mathfrak{pg} := \mathfrak{g}$.

Finally the symbol $\ni\!\!\!\!\supset$ stands for the semidirect sum of Lie superalgebras. If $\mathfrak{g} = \mathfrak{g}_1 \ni\!\!\!\!\supset \mathfrak{g}_2$ then $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is an ideal (resp. a subalgebra) of $\mathfrak{g}$.

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2. Nonlinear primitive Lie superalgebras

2.1. **Preliminary definitions and results.**

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a finite dimensional Lie superalgebra over the field of complex numbers and L_0 a subalgebra of L. The pair (L, L_0) is called a *transitive Lie superalgebra* if L_0 does not contain any nonzero ideal of L. A transitive Lie superalgebra is called *nonlinear primitive* if

(a) L_0 is a maximal subalgebra of L (*primitive*);
(b) the isotropy representation $\mathrm{ad}_{L_0}|_{L/L_0}$ of L_0 on L/L_0 has a nontrivial kernel, i.e., the subspace L_1 defined by

$$L_1 = \Big\{ x \in L_0 \mid [x, L] \subset L_0 \Big\}$$

is nonzero (*nonlinear*).

A transitive Lie superalgebra (L, L_0) is called *irreducible* if the isotropy representation is irreducible; it is immediate that any irreducible transitive Lie superalgebra is primitive. Two transitive Lie superalgebras (L, L_0) and (L', L'_0) are *isomorphic* if there exists a Lie superalgebra isomorphism $\varphi : L \to L'$ with $\varphi(L_0) = L'_0$.

The following result is proved in [21, Lemmata 1 and 2] (the proof is given in the Lie algebra case but it extends verbatim to the superalgebra case).

Lemma 2.1. *Let (L, L_0) be a nonlinear primitive Lie superalgebra. Then:*

(i) $L_0 \neq L_1$;
(ii) $[L_0, L_1] \subset L_1$;
(iii) $[L, L_1] \subset L_0$.

Furthermore:

(iv) *if J is a nonzero ideal of L, then $J + L_0 = L$;*
(v) *if J_1 and J_2 are ideals of L such that $J_1 \cap L_0 \neq (0)$ and $J_2 \neq (0)$, then $[J_1, J_2] \neq (0)$;*
(vi) *if J is a nonzero ideal of L, then $J \cap L_0 \neq (0)$;*
(vii) *two nonzero ideals of L are never in direct sum.*

Recall that a Lie superalgebra is called *semisimple* if its radical is zero, equivalently if it does not contain any nonzero abelian ideal.

Corollary 2.2. *Let (L, L_0) be a nonlinear primitive Lie superalgebra. Then L is a semisimple Lie superalgebra and it is not the direct sum of two nonzero ideals.*

In the Lie algebra case this immediately yields that L is simple, L_1 abelian and $L_2 = (0)$ (see [21, Lemma 3]). The corresponding statement is not true in the Lie superalgebra case and one has to separately consider simple and semisimple Lie superalgebras.

Semisimple Lie superalgebras are described in [14] (see also [9, Section 7]). We recall here only the facts that we need and refer to those texts for more details.

2.2. Semisimple Lie superalgebras.

Let n be a nonnegative integer and $V = \mathbb{C}^n$. We denote by $\Lambda = \Lambda(n) = \Lambda(V^*)$ the Grassmann algebra of V^* with its natural parity decomposition $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$ and by

$$W = W(n) = \mathrm{der}(\Lambda) = \Lambda \otimes \partial_V$$

the Lie superalgebra of supervector fields on the n-dimensional purely odd linear supermanifold. Here $\partial_V = \{\partial_\xi \mid \xi \in V^*\} \simeq V$ is the Lie superalgebra of constant supervector fields. There exists a direct sum decomposition

$$W = \partial_V \oplus \Lambda^+ \partial_V \ , \tag{2.2}$$

where Λ^+ is the (not unital) subalgebra of Λ generated by V^*.

Definition 2.3. A subalgebra of W is called *admissible* if it projects surjectively to ∂_V with respect to the decomposition (2.2).

Let $\mathfrak{s}_1, \dots, \mathfrak{s}_N$ be simple Lie superalgebras and $n_1, \dots, n_N$ nonnegative integers. We associate two Lie superalgebras

$$\mathfrak{s}^\Lambda = \bigoplus \mathfrak{s}_i \otimes \Lambda(n_i) \tag{2.3}$$

and

$$\begin{aligned}\mathfrak{s}_{\max} = \operatorname{der}(\mathfrak{s}^\Lambda) &= \bigoplus \operatorname{der}(\mathfrak{s}_i \otimes \Lambda(n_i)) \\ &= \bigoplus (\operatorname{der}(\mathfrak{s}_i) \otimes \Lambda(n_i) \ni \mathbf{1} \otimes W(n_i))\end{aligned}$$

and consider the natural projections

$$\pi_i : \mathfrak{s}_{\max} \to \mathfrak{s}_{\max}/\operatorname{Ker}\pi_i \simeq W(n_i) \qquad \text{with}$$

$$\operatorname{Ker}\pi_i = \bigoplus_{j \neq i} \operatorname{der}(\mathfrak{s}_j \otimes \Lambda(n_j)) \oplus (\operatorname{der}(\mathfrak{s}_i) \otimes \Lambda(n_i)) \ . \tag{2.4}$$

Note that $\mathfrak{s}^\Lambda$ is a subalgebra of $\mathfrak{s}_{\max}$ in a natural way. We call any subalgebra $\mathfrak{s}^\Lambda \subset \mathfrak{g} \subset \mathfrak{s}_{\max}$ an *intermediate subalgebra.*

Definition 2.4. An intermediate subalgebra $\mathfrak{g}$ is *admissible* if its projection $\pi_i(\mathfrak{g})$ is an admissible subalgebra of $W(n_i)$ for all $i = 1, \dots, N$.

Recall that the *socle* $\operatorname{soc}(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ is usually defined as the sum of all minimal ideals of $\mathfrak{g}$.

Theorem 2.5. [14] *An intermediate subalgebra $\mathfrak{g}$ is semisimple if and only if it is admissible and any semisimple Lie superalgebra is an admissible intermediate subalgebra, for some $\mathfrak{s}_1, \dots, \mathfrak{s}_N$ and $n_1, \dots, n_N$. Moreover if $\mathfrak{g}$ is semisimple:*

(i) *each $\mathfrak{s}_i \otimes \Lambda(n_i)$ is a minimal ideal of $\mathfrak{g}$ and $\operatorname{soc}(\mathfrak{g}) = \mathfrak{s}^\Lambda$;*
(ii) *$\operatorname{der}(\mathfrak{g})$ coincides with the normalizer $N_{\mathfrak{s}_{\max}}(\mathfrak{g})$ of $\mathfrak{g}$ in $\mathfrak{s}_{\max}$.*

The socle of a semisimple Lie superalgebra $\mathfrak{g}$ is a *characteristic ideal*, i.e. it is stable under all automorphisms and derivations of $\mathfrak{g}$ (it can be proved directly for the automorphisms and the even derivations, one needs point (ii) of Theorem 2.5 for the odd derivations). This leads to the following useful observations.

Remark 2.6. The $\mathfrak{s}_1, \dots, \mathfrak{s}_N$ and $n_1, \dots, n_N$ of Theorem 2.5 are uniquely determined by $\mathfrak{g}$. Indeed, since each $\mathfrak{s}_i \otimes \Lambda(n_i)$ is a minimal ideal, if

$$\operatorname{soc}(\mathfrak{g}) = \bigoplus_{i=1}^{N} \mathfrak{s}_i \otimes \Lambda(n_i) = \bigoplus_{i=1}^{M} \mathfrak{s}'_i \otimes \Lambda(m_i)$$

then $N = M$ and $\mathfrak{s}_i \otimes \Lambda(n_i) = \mathfrak{s}'_i \otimes \Lambda(m_i)$ up to a reordering of the indices. The claim follows from the fact that the radical of $\mathfrak{s}_i \otimes \Lambda(n_i)$ is $\mathfrak{s}_i \otimes \Lambda^+(n_i)$ and $\mathfrak{s}_i = \mathfrak{s}_i \otimes \Lambda(n_i)/\mathfrak{s}_i \otimes \Lambda^+(n_i)$.

Remark 2.7. Any isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of semisimple Lie superalgebras is the restriction $\phi = \varphi|_{\mathfrak{g}}$ of an automorphism φ of $\mathfrak{s}_{\max}$ with $\varphi(\mathfrak{g}) = \mathfrak{g}'$. Indeed $\mathrm{soc}(\mathfrak{g}) = \mathrm{soc}(\mathfrak{g}') = \mathfrak{s}^{\Lambda}$ and $\phi(\mathfrak{s}^{\Lambda}) = \mathfrak{s}^{\Lambda}$ by Remark 2.6; the claim follows then from the fact that ϕ is completely determined by the action on $\mathfrak{s}^{\Lambda}$ and this action, in turn, always extends to a unique automorphism of $\mathfrak{s}_{\max} = \mathrm{der}(\mathfrak{s}^{\Lambda})$.

We end this section with the following.

Proposition 2.8. *Let (L, L_0) be a nonlinear primitive Lie superalgebra. Then L is semisimple with socle of the form $\mathfrak{s}^{\Lambda} = \mathfrak{s} \otimes \Lambda(n)$ for some unique simple Lie superalgebra $\mathfrak{s}$ and nonnegative integer n. In particular L is an admissible intermediate subalgebra of the Lie superalgebra $\mathfrak{s}_{\max} = \mathrm{der}(\mathfrak{s}) \otimes \Lambda \ni 1 \otimes W(n)$.*

Proof. Semisimplicity is direct from Corollary 2.2. From Theorem 2.5, L is an intermediate admissible subalgebra and we assume $N \geq 2$, by contradiction. Then $s_1 \otimes \Lambda(n_1)$ and $\mathfrak{s}_2 \otimes \Lambda(n_2)$ are two ideals of L in direct sum, an absurd by Lemma 2.1. Hence $\mathfrak{s}^{\Lambda} = \mathfrak{s} \otimes \Lambda(n)$ and unicity follows from Remark 2.6. ☐

3. Filtrations and $\mathbb{Z}$-gradings of Lie superalgebras

From now on we will restrict to *irreducible* Lie superalgebras.

3.1. Preliminary results.

Let (L, L_0) be an *irreducible* transitive Lie superalgebra, i.e. the representation of L_0 on L/L_0 is irreducible. Following [32] we set $L_{-p} = L$ for every $p \geq 1$ and define subspaces L_p of L inductively:

$$L_p = \left\{ x \in L_{p-1} \mid [x, L] \subset L_{p-1} \right\} .$$

The collection $\{L_p\}_{p \geq -1}$ is a *filtration* of L, a chain of linear subspaces

$$L_p \subset L \ , \quad L_p = L_p \cap L_{\bar{0}} \oplus L_p \cap L_{\bar{1}} \ ,$$

which satisfies

$$L = \bigcup_p L_p,$$
$$0 = \bigcap_p L_p,$$

and, for all $p, q \in \mathbb{Z}$,

$$L_p \supset L_{p+1},$$
$$[L_p, L_q] \subset L_{p+q}.$$

The *associated $\mathbb{Z}$-graded Lie superalgebra* $\mathfrak{g} = \mathrm{gr}(L)$ is defined by

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^p \ , \quad \mathfrak{g}^p = L_p / L_{p+1} \ , \tag{3.5}$$

with the induced Lie bracket and parity decomposition.

Recall that a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ is called of *depth* $d(\mathfrak{g}) = d$ if $\mathfrak{g}^p = (0)$ for all $p < -d$ and $\mathfrak{g}^{-d} \neq (0)$. The Lie superalgebra (3.5) has the following basic properties:

(a) it is *transitive* i.e. if $x \in \mathfrak{g}^p$, $p \geq 0$, satisfies $[x, \mathfrak{g}^{-1}] = (0)$ then $x = 0$;
(b) it is of *depth* $d = 1$;
(c) it is *irreducible* i.e. the adjoint representation of $\mathfrak{g}^0$ on $\mathfrak{g}^{-1}$ is irreducible.

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ satisfying (a)-(c) is called a *transitive irreducible $\mathbb{Z}$-graded Lie superalgebra of depth* 1. Note that the corresponding zero-degree part is an irreducible linear Lie superalgebra $\mathfrak{g}^0 \subset \mathfrak{gl}(\mathfrak{g}^{-1})$.

From (i) of Lemma 2.1 the Lie superalgebra $\mathfrak{g}$ associated with a *nonlinear* (L, L_0) has also the additional property

(d) it is *nonlinear* i.e. $\mathfrak{g}^1 \neq (0)$

and this fact motivates the following.

Definition 3.1. A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ satisfying (a)-(d) is called a *transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebra of depth* 1.

There exists a kind of inverse construction $\mathfrak{g} \mapsto (L, L_0)$.

Proposition 3.2. *If $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ is a transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebra of depth 1 then $\mathfrak{g}$ is semisimple with socle $\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda(n)$ for some unique simple Lie superalgebra $\mathfrak{s}$ and nonnegative integer n. Moreover*

$$\mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_p \supset \cdots , \qquad \mathfrak{g}_p = \bigoplus_{i \geq p} \mathfrak{g}^i , \tag{3.6}$$

is the natural filtration on $\mathfrak{g}$ associated with the nonlinear irreducible transitive Lie superalgebra $(L, L_0) = (\mathfrak{g}, \mathfrak{g}_0 = \bigoplus_{i \geq 0} \mathfrak{g}^i)$.

Proof. By a direct check, the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is a nonlinear irreducible transitive Lie superalgebra with associated filtration (3.6) and the rest follows from Proposition 2.8. □

By Proposition 3.2 the first natural class of nonlinear irreducible transitive Lie superalgebras to be considered is given by the filtered Lie superalgebras directly associated with the $\mathbb{Z}$-graded Lie superalgebras of Definition 3.1; the main aim of the paper is the study of these latter.

We now need to recall the description of $\mathbb{Z}$-gradings of superalgebras. We deal only with the facts that we need and refer to e.g. [22] for more details.

3.2. **Gradings of a superalgebra.**

Let A be a (purely even) algebra. A *$\mathbb{Z}$-grading of A* (shortly a *grading*) is a direct sum decomposition of A in linear subspaces

$$A = \bigoplus_{p \in \mathbb{Z}} A^p \qquad \text{with} \quad A^p \cdot A^q \subset A^{p+q} . \tag{3.7}$$

The grading is called of *depth* d if $A^{-d} \neq (0)$ and $A^{-p} = (0)$ for all $p > d$. In other words, a grading is the eigenspace decomposition for a semisimple derivation $D \in \mathrm{der}(A)$, $D|_{A^p} = p\,\mathrm{Id}$ with integer eigenvalues. The operator D is called the *grading operator*. If $A = A_{\bar 0} \oplus A_{\bar 1}$ is a superalgebra, we assume that the grading is of the form $A^p = A^p_{\bar 0} \oplus A^p_{\bar 1}$ i.e. $D \in \mathrm{der}_{\bar 0}(A)$ is even.

Let $\mathrm{Aut}(A)$ be the Lie group of all automorphisms of A. The description of $\mathbb{Z}$-gradings of A up to automorphisms corresponds to the description of the grading operators up to conjugacy by $\mathrm{Aut}(A)$.

Definition 3.3. A subalgebra $\mathfrak{t}$ of $\mathrm{der}_{\bar 0}(A)$ is called *toric* if it is abelian and any $D \in \mathfrak{t}$ is semisimple. A maximal toric subalgebra $\mathfrak{t} \subset \mathrm{der}_{\bar 0}(A)$ is called a *maximal torus of* A.

Since $\mathrm{der}_{\bar 0}(A)$ is algebraic (it is the tangent Lie algebra of $\mathrm{Aut}(A)$, an algebraic subgroup of the group $\mathrm{GL}_{\bar 0}(A)$ of all even invertible transformations), then all maximal tori of A are conjugate by $\mathrm{Aut}^\circ(A)$ ([24]).

It follows that the description of $\mathbb{Z}$-gradings of a superalgebra A up to automorphisms reduces to find a maximal torus of A and its grading operators.

3.3. Gradings of $\Lambda = \Lambda(V^*)$ and $W = \mathrm{der}(\Lambda)$.

Let $V = \mathbb{C}^n$. We describe all the gradings of the Grassmann algebra $\Lambda = \Lambda(U)$, $U = V^*$ and the Lie superalgebra $W = \mathrm{der}(\Lambda) = \Lambda \otimes \partial_V \simeq \Lambda \otimes V$ of supervector fields, where $\partial_V \simeq V$ is the space of constant supervector fields. Let $U = U^{k_1} \oplus \cdots \oplus U^{k_m}$ be a grading of U, $\dim U^{k_i} = n_i$, defined by the grading operator

$$D = D_U = k_1\,\mathrm{Id}_{U^{k_1}} + \cdots + k_m\,\mathrm{Id}_{U^{k_m}}$$

(in this case just a semisimple linear endomorphism with integer eigenvalues).

Definition 3.4. The spectrum $\vec{k} := (k_1, \ldots, k_m)$ of D is called the *type* of the grading.

We always assume that $n_i \neq 0$ and $k_1 < \cdots < k_m$. The operator D naturally acts as a grading operator on V, Λ and W and defines the grading

$$V = V^{-k_1} \oplus \cdots \oplus V^{-k_m}\ , \qquad V^{-k_i} = (U^{k_i})^*\ ,$$

of V and the gradings $\Lambda = \bigoplus \Lambda^p$ and $W = \bigoplus W^p$ of Λ and, respectively, W given by

$$\begin{aligned} \Lambda^p &= \bigoplus_{p_1k_1+\cdots+p_mk_m=p} \Lambda^{p_1}(U^{k_1}) \cdot \Lambda^{p_2}(U^{k_2}) \cdots \Lambda^{p_m}(U^{k_m})\ , \\ W^q &= \bigoplus_{p_i - k_i = q} \Lambda^{p_i} \partial_{V^{-k_i}}\ . \end{aligned} \tag{3.8}$$

We denote the corresponding depths by $d(\Lambda)$ and $d(W)$. By a little abuse of notation, we also say that these gradings are of *type* $\vec{k}$. The grading of type $\vec{k} = (1)$, defined by the identity endomorphism $D = \mathrm{Id}$, is usually called the *principal grading*; in this case the degree $\deg a$ of $a \in \Lambda$ coincides with the degree of a as an exterior form.

Proposition 3.5. *The space $\mathfrak{t}_{\mathrm{W}}$ of diagonal matrices in $\mathfrak{gl}(V) \subset W$ is a maximal torus of W and all maximal tori are conjugated by* $\mathrm{Aut}(\Lambda)$. *Any grading of Λ and W is conjugated by* $\mathrm{Aut}(\Lambda)$ *to a grading of type $\vec{k}$.*

Proof. First note that the center $Z(W)$ of W is trivial and that $\mathrm{der}(W) = W$ (one directly checks the statement if $n = 1$ whereas, for $n \geq 2$, $W(n)$ is simple and [14, Prop. 5.1.2] applies). The space $\mathfrak{t}_{\mathrm{W}}$ is a subspace of $\mathrm{der}_{\overline{0}}(\Lambda)$; it is a toric subalgebra, since the action of any of its elements on Λ is semisimple. One can then apply e.g. [16, Lemma 1] to the principal grading of W and get that $\mathfrak{t}_W$ is a maximal torus of W (hence of Λ too, since $\mathrm{der}_{\overline{0}}(\Lambda) = \mathrm{der}_{\overline{0}}(W)$). All maximal tori are conjugated, by the discussion at the end of Section 3.2.

The last claim follows from the fact that the collection of the supervector fields $H_\alpha = \xi^\alpha \partial_{\xi^\alpha}$, $\alpha = 1, \ldots, n$, is a basis of $\mathfrak{t}_{\mathrm{W}}$ and the grading operators in $\mathfrak{t}_{\mathrm{W}}$ are the linear combinations of the H_α's with integer coefficients. □

Since $\deg 1 = 0$, the depth $d(\Lambda) \geq 0$ and is given by

$$d(\Lambda) = \begin{cases} 0 \ \text{ if } \ k_1 \geq 0, \\ \sum_{k_i < 0} \ |k_i| n_i \ \text{ if } \ k_1 < 0. \end{cases}$$

The depth $d(W) = k_m + d(\Lambda) \geq k_m$. This implies the following.

Lemma 3.6. *i) A grading of Λ of type $\vec{k}$ has depth $d(\Lambda) = 1$ if and only if $\vec{k} = (-1, k_2, \ldots, k_m)$ and $\dim U^{-1} = 1$, i.e. $U^{-1} = \mathbb{C}\xi$. In this case $d(W) = k_m + 1$.*
ii) A grading of Λ of type $\vec{k}$ has depth $d(\Lambda) = 0$ if and only if $k_1 \geq 0$. Moreover $d(W) = 0$ if and only if $\vec{k} = (0)$ (trivial grading) whereas $d(W) = 1$ if and only if $\vec{k} = (0, 1)$ or $\vec{k} = (1)$.

3.4. **Gradings of a semisimple Lie superalgebra with socle $\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$.**

The following result reduces the description of graded semisimple Lie superalgebras $\mathfrak{g}$ with socle $\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$, where $\mathfrak{s}$ is a simple Lie superalgebra, to the description of graded admissible subalgebras $\mathfrak{s}^\Lambda \subset \mathfrak{g} = \mathfrak{s}_{\max}$ of the Lie superalgebra $\mathfrak{s}_{\max} = \mathrm{der}(\mathfrak{s}) \otimes \Lambda \ni \mathbf{1} \otimes W$ with a fixed grading. To state the result we note that any grading of $\mathfrak{s}$ induces a natural grading on its algebra $\mathrm{der}(\mathfrak{s})$ of derivations.

Proposition 3.7. *Any grading of a semisimple Lie superalgebra $\mathfrak{g}$ with socle $\mathfrak{s} \otimes \Lambda$ is, up to automorphisms in* $\mathrm{Aut}(\mathfrak{s}_{\max})$, *induced by the grading of $\mathfrak{s}_{\max}$*

$$\mathfrak{s}_{\max} = \bigoplus \mathfrak{s}^p_{\max} \ , \qquad \mathfrak{s}^p_{\max} = \bigoplus_{i+j=p} (\mathrm{der}^i(\mathfrak{s}) \otimes \Lambda^j) \oplus W^p \ ,$$

generated by a grading of $\mathfrak{s}$ and the gradings of Λ and W associated with a given grading of U of type $\vec{k} = (k_1, \ldots, k_m)$.

Proof. We first prove that $\mathfrak{t} = \mathfrak{t}_{\mathfrak{s}} \oplus \mathfrak{t}_{\mathrm{W}}$ is a maximal torus of $\mathfrak{s}_{\max}$, where $\mathfrak{t}_{\mathfrak{s}}$ is a maximal torus of $\mathfrak{s}$ (that is a maximal toric subalgebra of $\operatorname{der}(\mathfrak{s})$) and $\mathfrak{t}_{\mathrm{W}}$ is the space of diagonal matrices in W.

It is clearly toric, as $[\mathfrak{t}_{\mathfrak{s}}, \mathfrak{t}_{\mathrm{W}}] = (0)$ and the action of any $x \in \mathfrak{t}$ is semisimple on $\mathfrak{s}^{\Lambda}$ and hence on $\mathfrak{s}_{\max} = \operatorname{der}(\mathfrak{s}^{\Lambda})$. Now $\mathfrak{t}_{\mathrm{W}}$ is a maximal torus of W by Proposition 3.5 and $\mathfrak{t}_{\mathfrak{s}}$ is maximal toric in $\operatorname{der}(\mathfrak{s}) \otimes \Lambda$. Indeed if $x \in \operatorname{der}(\mathfrak{s}) \otimes \Lambda$ is an even element with $[x, \mathfrak{t}_{\mathfrak{s}}] = (0)$ then $x = x^o + x^+ \in \operatorname{der}(\mathfrak{s}) \oplus \operatorname{der}(\mathfrak{s}) \otimes \Lambda^+$ with $[x^o, \mathfrak{t}_{\mathfrak{s}}] = [x^+, \mathfrak{t}_{\mathfrak{s}}] = (0)$ and its action on $\mathfrak{s} \otimes \Lambda^n \simeq \mathfrak{s}$ reduces to the natural action of x^o. It follows that if x is semisimple then x^o is semisimple, $x^o \in \mathfrak{t}_{\mathfrak{s}}$ and $x^+ = 0$.

This yields that $\mathfrak{t}$ is a maximal torus of $\mathfrak{s}_{\max}$ and that any grading operator $D \in \mathfrak{t}$ of $\mathfrak{s}_{\max}$ decomposes in $D = D_{\mathfrak{s}} + D_{\mathrm{W}}$ where $D_{\mathfrak{s}} \in \mathfrak{t}_{\mathfrak{s}}$ (resp. $D_{\mathrm{W}} \in \mathfrak{t}_{\mathrm{W}}$) is a grading operator of $\mathfrak{s}$ (resp. a grading of type $\vec{k} = (k_1, \dots, k_m)$).

There is a natural 1-1 correspondence between the grading operators of $\mathfrak{g}$ and those of $\mathfrak{s}_{\max}$ preserving $\mathfrak{g}$. First of all note that the grading operators of $\mathfrak{s}_{\max}$ and those of $\mathfrak{s}^{\Lambda}$ are in a natural 1-1 correspondence, as $\operatorname{der}(\mathfrak{s}_{\max}) = \mathfrak{s}_{\max} = \operatorname{der}(\mathfrak{s}^{\Lambda})$. Finally any grading operator of $\mathfrak{g}$ satisfies

$$D \in \operatorname{der}_{\bar{0}}(\mathfrak{g}) = N_{\mathfrak{s}_{\max}}(\mathfrak{g})_{\bar{0}} \subset (\mathfrak{s}_{\max})_{\bar{0}}$$

by Theorem 2.5 and $D|_{\mathfrak{s}^{\Lambda}} : \mathfrak{s}^{\Lambda} \to \mathfrak{s}^{\Lambda}$ is a grading operator of $\mathfrak{s}^{\Lambda}$ which extends to a unique grading operator of $\mathfrak{s}_{\max}$. □

3.5. Gradings of the Lie superalgebra $\operatorname{der}(\mathfrak{s})$ with $\mathfrak{s}$ simple.

The main aim of this section is to study in detail the case where the socle is a simple Lie superalgebra. In other words we have $n = 0$, $V = (0)$, $\Lambda = 1$ and $W = (0)$. An important rôle is played by the gradings of the Lie superalgebra $\operatorname{der}(\mathfrak{s})$ of derivations of a simple Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$.

3.5.1. *Preliminary results.*

Finite-dimensional simple Lie superalgebras are classified in [14] and split into two main families: *classical* superalgebras, for which the adjoint action of $\mathfrak{s}_{\bar{0}}$ on $\mathfrak{s}_{\bar{1}}$ is completely reducible, and *Cartan* superalgebras $W(n)$ (for $n \geq 3$), $S(n)$ (for $n \geq 4$), $\widetilde{S}(n)$ (for $n \geq 4$ and even), $H(n)$ (for $n \geq 5$), that is finite-dimensional superalgebras analogue to simple Lie algebras of vector fields.

Remark 3.8. The simple Lie superalgebras $W(2)$, $S(3)$, $\widetilde{S}(2)$ and $H(4)$ are isomorphic to the classical superalgebras $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$, $\mathfrak{spe}(3)$, $\mathfrak{osp}(1|2)$ and $\mathfrak{psl}(2|2)$, respectively. In our conventions, they are not Cartan.

Classical superalgebras in turn split into the so-called *basic* superalgebras $\mathfrak{sl}(m+1|n+1)$ (for $m < n$), $\mathfrak{psl}(n+1|n+1)$ (for $n \geq 1$), $\mathfrak{osp}(2m+1|2n)$ (for $n \geq 1$), $\mathfrak{osp}(2|2n-2)$ (for $n \geq 3$), $\mathfrak{osp}(2m|2n)$ (for $m \geq 2$, $n \geq 1$), $\mathfrak{osp}(4|2;\alpha)$ (for $\alpha \neq 0, \pm 1, -2, -\frac{1}{2}$), $\mathfrak{ab}(3)$, $\mathfrak{ag}(2)$, for which there exists a non-degenerate even invariant supersymmetric bilinear form $B : \mathfrak{s} \otimes \mathfrak{s} \to \mathbb{C}$, and two *strange* families $\mathfrak{spe}(n)$ (for $n \geq 3$) and $\mathfrak{psq}(n)$ (for $n \geq 3$). The form B is unique up to constant and it coincides with the Killing form of $\mathfrak{s}$, except for the cases $\mathfrak{psl}(n+1|n+1)$, $\mathfrak{osp}(2m+2|2m)$ and $\mathfrak{osp}(4|2;\alpha)$. The continuous family $\mathfrak{osp}(4|2;\alpha)$ consists of deformations of $\mathfrak{osp}(4|2)$ and two Lie superalgebras $\mathfrak{osp}(4|2;\alpha)$ and $\mathfrak{osp}(4|2;\alpha')$ are isomorphic if and only if α and α' lie on the same orbit under the action of the permutation group $\mathfrak{S}_3$ generated by $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto (-1-\alpha)$.

Table 1 gives the description of the even Lie subalgebra $\mathfrak{s}_{\bar{0}}$ of $\mathfrak{s}$ and its representation on $\Pi(\mathfrak{s}_{\bar{1}})$ (therein $\mathbb{S}$ denotes the spin module of $\mathfrak{spin}(7)$).

$\mathfrak{s}$	$\mathfrak{s}_{\bar{0}}$	$\Pi(\mathfrak{s}_{\bar{1}})$
$\mathfrak{sl}(m+1\|n+1)$ $m<n$	$\mathfrak{sl}(m+1) \oplus \mathfrak{sl}(n+1) \oplus Z$	$\mathbb{C}^{m+1} \otimes (\mathbb{C}^{n+1})^* \oplus (\mathbb{C}^{m+1})^* \otimes \mathbb{C}^{n+1}$
$\mathfrak{psl}(n+1\|n+1)$ $n \geq 1$	$\mathfrak{sl}(n+1) \oplus \mathfrak{sl}(n+1)$	$\mathbb{C}^{n+1} \otimes (\mathbb{C}^{n+1})^* \oplus (\mathbb{C}^{n+1})^* \otimes \mathbb{C}^{n+1}$
$\mathfrak{osp}(2m+1\|2n)$ $n \geq 1$	$\mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$	$\mathbb{C}^{2m+1} \otimes \mathbb{C}^{2n}$
$\mathfrak{osp}(2\|2n-2)$ $n \geq 3$	$\mathfrak{so}(2) \oplus \mathfrak{sp}(2n-2)$	$\mathbb{C}^{2} \otimes \mathbb{C}^{2n-2}$
$\mathfrak{osp}(2m\|2n)$ $m \geq 2, n \geq 1$	$\mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$	$\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$
$\mathfrak{osp}(4\|2;\alpha)$ $\alpha \neq 0, \pm 1, -2, -1/2$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$\mathfrak{ab}(3)$	$\mathfrak{spin}(7) \oplus \mathfrak{sl}(2)$	$\mathbb{S} \otimes \mathbb{C}^2$
$\mathfrak{ag}(2)$	$\mathrm{G}_2 \oplus \mathfrak{sl}(2)$	$\mathbb{C}^7 \otimes \mathbb{C}^2$
$\mathfrak{spe}(n)$ $n \geq 3$	$\mathfrak{sl}(n)$	$S^2(\mathbb{C}^n) \oplus \Lambda^2((\mathbb{C}^n)^*)$
$\mathfrak{psq}(n)$ $n \geq 3$	$\mathfrak{sl}(n)$	$\mathrm{ad}(\mathfrak{sl}(n))$

TABLE 1. Classical superalgebras $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$.

A direct inspection of [14, Prop. 5.1.2] and its proof implies the following.

Proposition 3.9. *Let $\mathfrak{s}$ be a simple Lie superalgebra. Then* $\mathrm{der}(\mathfrak{s})$ *admits a semidirect decomposition* $\mathrm{der}(\mathfrak{s}) = \mathfrak{s} \ni \mathrm{out}(\mathfrak{s})$ *for a subalgebra* $\mathrm{out}(\mathfrak{s})$ *of outer derivations and a maximal torus of $\mathfrak{s}$ (that is a maximal toric subalgebra of* $\mathrm{der}(\mathfrak{s})$*) is conjugated to a maximal torus of the form $\mathfrak{t}_{\mathfrak{s}} = \mathfrak{h} \oplus \mathfrak{t}_o$ for a maximal toric subalgebra $\mathfrak{h}$ in $\mathfrak{s}_{\bar{0}}$ and a maximal toric subalgebra $\mathfrak{t}_o$ in* $\mathrm{out}_{\bar{0}}(\mathfrak{s})$*. In particular there is a natural 1-1 correspondence between gradings of $\mathfrak{s}$, gradings of* $\mathrm{der}(\mathfrak{s})$ *and grading operators $D \in \mathfrak{t}_{\mathfrak{s}}$. Moreover* $[\mathfrak{h}, \mathrm{out}(\mathfrak{s})] = (0)$ *in all cases and $\mathfrak{t}_o$ has dimension one if $\mathfrak{s} = \mathfrak{psl}(n+1|n+1)$, $\mathfrak{spe}(n)$, $S(n)$, $H(n)$ and it is trivial for all other simple Lie superalgebras.*

We note that the subalgebra $\mathrm{out}(\mathfrak{s})$ is always stable under any grading operator $D \in \mathfrak{t}_{\mathfrak{s}}$ and hence graded. The cases where $\mathrm{out}(\mathfrak{s})$ is nontrivial are summarized in Table 2, for the reader's convenience (therein we denote by T a two-dimensional solvable Lie superalgebra of dimension $(1,1)$ if n is odd, respectively of dimension (2,0) is n is even).

$\mathfrak{s}$	$\mathfrak{psl}(n+1\|n+1)$ $n \geq 2$	$\mathfrak{psl}(2\|2)$	$S(n)$ $n \geq 4$	$\mathfrak{spe}(n)$ $n \geq 3$	$\mathfrak{psq}(n)$ $n \geq 3$	$H(n)$ $n \geq 5$
$\mathrm{out}(\mathfrak{s})$	$\mathbb{C}$	$\mathfrak{sl}(2)$	$\mathbb{C}$	$\mathbb{C}$	$\Pi(\mathbb{C})$	T

TABLE 2. The algebras of outer derivations of simple Lie superalgebras.

Recall that a representation of a Lie superalgebra is called *irreducible of G-type* if it does not admit any nontrivial submodule whether or not $\mathbb{Z}_2$-graded, see [3]. The following is a basic but useful result.

Lemma 3.10. *Let $\mathfrak{s}$ be a simple Lie superalgebra. If $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ is a grading of depth one and* $\mathrm{der}(\mathfrak{s}) = \bigoplus \mathrm{der}^p(\mathfrak{s})$ *the associated grading of* $\mathrm{der}(\mathfrak{s})$*, then:*

i) *if $x \in \mathrm{der}^p(\mathfrak{s})$, $p \geq 0$, satisfies $[x, \mathfrak{s}^{-1}] = (0)$ then $x = 0$;*
ii) *$\mathfrak{s}^0$ and $\mathfrak{s}^1$ are nonzero;*
iii) *the adjoint action of $\mathfrak{s}^0$ on $\mathfrak{s}^{-1}$ is irreducible of G-type;*
iv) *the depth $d(\mathrm{der}(\mathfrak{s})) \geq d(\mathfrak{s})$ and if $D \in \mathfrak{h}$ then $d(\mathrm{der}(\mathfrak{s})) = d(\mathfrak{s})$;*
v) *if $x \in \mathfrak{s}^{-1}$ satisfies $[\mathfrak{s}^0, x] = (0)$ then $x = 0$.*

Proof. (i) As $\mathfrak{s}^{-1} = \mathfrak{s}^{-1}_{\bar{0}} \oplus \mathfrak{s}^{-1}_{\bar{1}}$ is a subspace of $\mathfrak{s}$, it is enough to consider an homogeneous x. If $x \in \mathfrak{s}^p$, the condition $[x, \mathfrak{s}^{-1}] = (0)$ implies that the ideal $< x >$ generated by x in $\mathfrak{s}$ is nonnegatively graded, hence trivial and $x = 0$.

If $x \in \mathrm{out}^p(\mathfrak{s})$ satisfies $[x, \mathfrak{s}^{-1}] = (0)$ then for all $y \in \mathfrak{s}^0$ and $z \in \mathfrak{s}^{-1}$ one has $[y, z] \in \mathfrak{s}^{-1}$ and $[[x, y], z] = [x, [y, z]] \pm [[x, z], y] = 0$. It follows that $[x, y] \in \mathfrak{s}^p$ is zero and that $[x, \mathfrak{s}^p] = (0)$ for any $p \geq 0$, by a simple induction process. Summarizing the adjoint action of x on $\mathfrak{s}$ is trivial and $x = 0$.

(ii) First of all $\mathfrak{s}^0 \neq (0)$ otherwise $\mathfrak{s}^p = (0)$ for any $p \geq 0$ and $\mathfrak{s} = \mathfrak{s}^{-1}$ is abelian, by point (i). Similarly $\mathfrak{s}^1 \neq (0)$ otherwise $\mathfrak{s}^p = (0)$ for any $p \geq 1$ and $\mathfrak{s}^{-1}$ is a nontrivial ideal of $\mathfrak{s} = \mathfrak{s}^{-1} \oplus \mathfrak{s}^0$.

(iii) Let $\mathfrak{m}$ be a not necessarily $\mathbb{Z}_2$-graded $\mathfrak{s}^0$-submodule of $\mathfrak{s}^{-1}$ and

$$< \mathfrak{m} >= \mathfrak{m} \oplus \mathrm{span}\left\{ [\mathfrak{s}^{p_1}, [\mathfrak{s}^{p_2}, [\ldots, [\mathfrak{s}^{p_q}, \mathfrak{m}] \ldots]]] \mid q, p_i > 0 \right\}$$

the ideal generated by $\mathfrak{m}$ in $\mathfrak{s}$; it is $\mathbb{Z}$-graded with $< \mathfrak{m} >^{-1} = \mathfrak{m}$. The fact that a simple Lie superalgebra does not admit any nontrivial ideal whether or not $\mathbb{Z}_2$-graded (see e.g. [27]) forces $\mathfrak{m} = \mathfrak{s}^{-1}$ or $\mathfrak{m} = (0)$.

(iv) it is clear from Proposition 3.9.

(v) a direct consequence of (iii). □

By Proposition 3.9, Lemma 3.10 and the fact that $\mathfrak{s}^{-1} \subset \mathrm{der}^{-1}(\mathfrak{s})$ is stable under the adjoint action of $\mathrm{der}^0(\mathfrak{s})$ one gets the following.

Proposition 3.11. *Let $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ be a grading of depth one of a simple Lie superalgebra. Any graded subalgebra $\mathfrak{s} \subset \mathfrak{g} \subset \mathrm{der}(\mathfrak{s})$ is of the form $\mathfrak{g} = \mathfrak{s} \ni F$ for some graded subalgebra $F = \mathfrak{g} \cap \mathrm{out}(\mathfrak{s})$ of $\mathrm{out}(\mathfrak{s})$ and it is a transitive non-linear $\mathbb{Z}$-graded Lie superalgebra. Moreover it is of depth one and irreducible if and only if the depth $d(F) \leq 0$ (i.e. F is graded in nonnegative degrees).*

The classification of depth one gradings of the simple Lie superalgebras $\mathfrak{s}$ was developed in [16, 29] (see also e.g. [25]). We now recall it and, in turn, also describe the associated gradings of $\mathrm{out}(\mathfrak{s})$ and $\mathrm{der}(\mathfrak{s})$.

3.5.2. *Basic superalgebras.*

Basic Lie superalgebras admit a convenient description in terms of root systems, Cartan matrices and Dynking diagrams; we recall here only the facts that we need and refer for more details to [14, 15, 28, 6].

Let $\mathfrak{s}$ be a basic superalgebra, $\mathfrak{t}_{\mathfrak{s}} = \mathfrak{h} \oplus \mathfrak{t}_o$ a maximal torus of $\mathfrak{s}$ ($\mathfrak{t}_o \subset \mathrm{out}_{\bar{0}}(\mathfrak{s})$ is trivial with the exception of $\mathfrak{psl}(n+1|n+1)$) and $\Delta = \Delta(\mathfrak{s}, \mathfrak{t}_{\mathfrak{s}})$ the associated root system. Then $\mathfrak{s}_{\bar{0}}$ and $\mathfrak{s}_{\bar{1}}$ decompose into the direct sum of root spaces $\mathfrak{s}^{\alpha}$ and a root α is called *even* (resp. *odd*) if $\mathfrak{s}^{\alpha}_{\bar{0}}$ (resp. $\mathfrak{s}^{\alpha}_{\bar{1}}$) is nonzero. Every root is either even or odd and the root spaces are all one-dimensional (for $\mathfrak{s} = \mathfrak{psl}(2|2)$ this follows from the fact that $\mathfrak{t}_o$ is non-trivial). Many properties of root systems of Lie algebras remain true for basic Lie superalgebras, see [15, Proposition 5.3]. In particular, any decomposition $\Delta = \Delta^+ \cup -\Delta^+$ into positive and negative roots determines a system

$$\Sigma = \{\alpha_1, \ldots, \alpha_r\}$$

of simple positive roots and every positive root $\alpha \in \Delta^+$ can be written as a sum

$$\alpha = \sum_{i=1}^{r} b_i \alpha_i$$

with non-negative integer coefficients b_i.

The Weyl group of the reductive Lie algebra $\mathfrak{s}_{\bar{0}}$ acts on the set of simple root systems. In contrast with the Lie algebra case this action is not transitive, and hence different simple root systems of the same basic Lie superalgebra may not be conjugated. To each orbit of the Weyl group one can associate a Dynkin diagram as follows (they were first introduced in [14, 28]; we will use the slightly different conventions given by [6]).

Consider a Cartan matrix (a_{ij}) of order r associated to Σ, see [14, 6]. Each simple root α_i corresponds to a node which is colored *white* ○ if α is even (in this case $a_{ii} = 2$), *gray* ◍ if α is odd and B-isotropic (in this case $a_{ii} = 0$), or *black* ● if α is odd and non-isotropic (in this case $a_{ii} = 1$).

The i-th and j-th nodes of the diagram are not joined if $a_{ij} = a_{ji} = 0$, otherwise they are joined by $\max(|a_{ij}|, |a_{ji}|)$-edges with an arrow pointing from α_i to α_j if $|a_{ij}| < |a_{ji}|$. If $\mathfrak{g} = \mathfrak{osp}(4|2; \alpha)$ the Cartan matrix is integer if and only if α is an integer; in this case we illustrate just with $\alpha = 2$.

Finally, we mark the i-th node with the corresponding Dynkin mark, that is the coefficient m_i of the expression of the highest root as sum of simple roots

$$\alpha_{\max} = \sum_{i=1}^{r} m_i \alpha_i \ .$$

The list of all possible Dynkin diagrams associated to basic Lie superalgebras is contained in [6, Tables 1-5].

Now, setting $\deg \alpha_i = \lambda_i$ for some nonnegative integer λ_i and extending the definition to all roots by

$$\deg(\sum_{i=1}^{r} b_i \alpha_i) = \sum_{i=1}^{r} b_i \deg(\alpha_i) \ ,$$

one gets the grading of $\mathfrak{s}$:

$$\mathfrak{s}^0 = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Delta \\ \deg\alpha=0}} \mathfrak{s}^\alpha \ , \qquad \mathfrak{s}^p = \bigoplus_{\substack{\alpha\in\Delta \\ \deg\alpha=p}} \mathfrak{s}^\alpha \quad \text{for} \quad p \neq 0 \ . \tag{3.9}$$

By [16, Theorem 1 & Remark 3], all possible gradings of $\mathfrak{s}$ are equivalent to one of this form, for some choice of Σ. The depth $d(\mathfrak{s})$ of $\mathfrak{s}$ is equal to the degree of the highest root

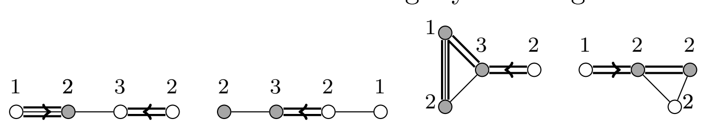
$$\deg(\alpha_{\max}) = \sum_{i=1}^{r} b_{i,\max} \deg(\alpha_i)$$

and coincides also with the height ℓ of $\mathfrak{s}$. Note that the Lie superalgebra (3.9) has depth one if and only if all $\lambda_j = 0$ with the exception of a simple root α_i, called the *crossed root*, which satisfies $m_i = 1$ and $\lambda_i = 1$.

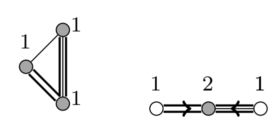
The subalgebra $\mathfrak{s}^0$ is the direct sum of a center and a Lie superalgebra whose Cartan matrix is obtained from the Cartan matrix of $\mathfrak{s}$ by removing all rows and columns relative to the crossed root.

Table 3 displays all depth one gradings of the the basic Lie superalgebras (see [29, 16, 25]). The symbol ◑ indicates a node which can be either white or gray and V_μ stands for the irreducible module for (the semisimple part of) the Lie superalgebra $\mathfrak{s}^0$ with the highest weight μ and even highest vector.

We remark that some gradings in Table 3 are listed more than once. We also note that the gradings of $\mathfrak{ab}(3)$ and $\mathfrak{osp}(4|2;\alpha)$ admit additional isomorphic presentations in terms of the following Dynkin diagrams of $\mathfrak{ab}(3)$



and of $\mathfrak{osp}(4|2;\alpha)$



These presentations are not displayed in Table 3.

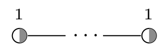
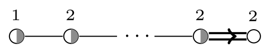
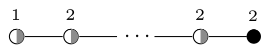
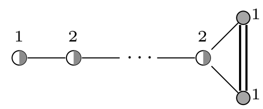
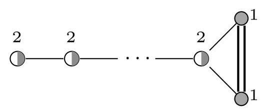
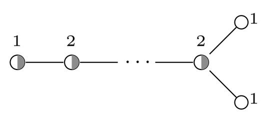
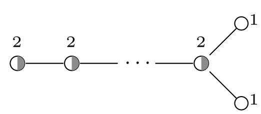
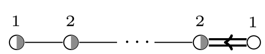
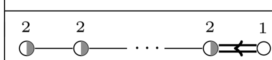
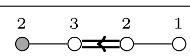
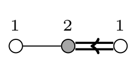
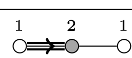
$\mathfrak{s}$	Dynkin diagram	order (a_{ij})	parity of the number of ●	crossed root	$\mathfrak{s}^{-1} = (\mathfrak{s}^1)^*$	$\mathfrak{s}^0$
$\mathfrak{sl}(m+1\|n+1)$ $m<n$		$m+n+1$	any	any	$\mathbb{C}^{m+1-p\|n+1-q} \otimes (\mathbb{C}^{p\|q})^*$	$\mathfrak{s}(\mathfrak{gl}(m+1-p\|n+1-q) \oplus \mathfrak{gl}(p\|q))$
$\mathfrak{psl}(m+1\|n+1)$ $m=n\neq 0$						$\mathfrak{ps}(\mathfrak{gl}(m+1-p\|n+1-q) \oplus \mathfrak{gl}(p\|q))$
$\mathfrak{osp}(2m+1\|2n)$ $m,n\geq 1$		$m+n$ $(m+n\geq 3)$	0	first	$\mathbb{C}^{2m-1\|2n}$	$\mathfrak{cosp}(2m-1\|2n)$
		m+n	1	first	$\mathbb{C}^{2m-1\|2n}$	$\mathfrak{cosp}(2m-1\|2n)$
$\mathfrak{osp}(2m,2n)$ $m,n\geq 1$ $m+n\geq 3$		$m+n$	1	first last	$\mathbb{C}^{2m-2\|2n}$ $\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{cosp}(2m-2\|2n)$ $\mathfrak{gl}(m\|n)$
		$m+n$	0	last	$\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{gl}(m\|n)$
		m+n	0	first last	$\mathbb{C}^{2m-2\|2n}$ $\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{cosp}(2m-2\|2n)$ $\mathfrak{gl}(m\|n)$
		$m+n$	1	last	$\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{gl}(m\|n)$
		m+n	1	first last	$\mathbb{C}^{2m-2\|2n}$ $\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{cosp}(2m-2\|2n)$ $\mathfrak{gl}(m\|n)$
		m+n	0	last	$\Lambda^2(\mathbb{C}^{m\|n})$	$\mathfrak{gl}(m\|n)$
$\mathfrak{ab}(3)$		4	1	last	$V_{-\epsilon_1+\delta_1+\delta_2}$	$\mathfrak{cosp}(2\|4)$
$\mathfrak{osp}(4\|2;\alpha)$ $\alpha\neq 0,\pm 1,-2,-1/2$		3	1	first last	$V_{-\alpha^{-1}\epsilon_1}$ $V_{-\alpha\epsilon_1}$	$\mathfrak{gl}(1\|2)$
		3	1	last	$V_{(1+\alpha)^{-1}\epsilon_1}$	$\mathfrak{gl}(1\|2)$

TABLE 3. The depth one gradings of the basic Lie superalgebras.

Now $\mathfrak{s} = \mathrm{der}(\mathfrak{s})$ for all basic Lie superalgebras, except $\mathfrak{s} = \mathfrak{psl}(n+1|n+1)$.

If $n \geq 2$ then $\mathrm{der}(\mathfrak{s}) \simeq \mathfrak{pgl}(n+1|n+1)$. More precisely $\mathrm{out}(\mathfrak{s}) = \mathfrak{t}_o$ is generated by an even element Z which acts trivially on $\mathfrak{s}_{\bar{0}}$ and with eigenvalues ± 1 on the two components of $\mathfrak{s}_{\bar{1}}$ (recall Table 1). In particular Z has degree zero for all gradings of $\mathfrak{s}$ and $\mathrm{der}^0(\mathfrak{s}) \simeq \mathfrak{p}(\mathfrak{gl}(n+1-p|n+1-q) \oplus \mathfrak{gl}(p|q))$.

If $\mathfrak{s} = \mathfrak{psl}(2|2)$ then $\mathrm{der}(\mathfrak{s}) \simeq \mathfrak{s} \ni \mathfrak{sl}(2)$ where $\mathfrak{sl}(2)$ is generated by an element Z as above and the two nilpotent even derivations $Z_\pm$ given by

$$\begin{aligned}
&Z_\pm(\mathfrak{s}_{\bar{0}}) = Z_+(\mathbb{C}^2 \otimes (\mathbb{C}^2)^*) = Z_-((\mathbb{C}^2)^* \otimes \mathbb{C}^2) = (0)\ ,\\
&Z_+ : (\mathbb{C}^2)^* \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes (\mathbb{C}^2)^* \quad \text{is an isomorphism of } \mathfrak{s}_{\bar{0}} - \text{modules}\ ,\\
&Z_-\big|_{\mathbb{C}^2 \otimes (\mathbb{C}^2)^*} = (Z_+\big|_{(\mathbb{C}^2)^* \otimes \mathbb{C}^2})^{-1} : \mathbb{C}^2 \otimes (\mathbb{C}^2)^* \longrightarrow (\mathbb{C}^2)^* \otimes \mathbb{C}^2\ .
\end{aligned}$$

A direct computation which uses an explicit decomposition of $\mathfrak{psl}(2|2)$ in root spaces gives Table 4 with the gradings of $\mathfrak{psl}(2|2)$ and $\mathfrak{sl}(2) = \langle Z_+, Z, Z_- \rangle$.

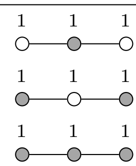
$\mathfrak{s}$	Dynkin diagram	Grading	$\deg(Z)$	$\deg(Z_+) = -\deg(Z_-)$
$\mathfrak{psl}(2\vert 2)$		$\deg(\alpha_i) = \lambda_i \geq 0$	0	$\lambda_1 + 2\lambda_2 + \lambda_3$ $\lambda_1 - \lambda_3$ $\lambda_1 + \lambda_3$

TABLE 4. The gradings of $\mathrm{der}(\mathfrak{s}) = \mathfrak{psl}(2|2) \ni \mathfrak{sl}(2)$.

Proposition 3.12. *All transitive nonlinear irreducible graded Lie subalgebras* $\mathfrak{s} \subset \mathfrak{g} \subset \mathrm{der}(\mathfrak{s})$ *of depth* 1, $\mathfrak{s} = \mathfrak{psl}(2|2)$, *are of the form* $\mathfrak{g} = \mathfrak{s} \ni F$ *where* $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ *is in Table 5 and* F *is a nonnegatively graded subalgebra of* $\mathfrak{sl}(2)$.

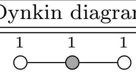
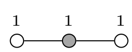
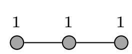
Dynkin diagram	crossed root	$\mathfrak{s}^{-1} = (\mathfrak{s}^1)^*$	$\mathfrak{s}^0$	$\mathfrak{sl}(2)^0$	$\mathfrak{sl}(2)^1$	$\mathfrak{sl}(2)^2$
	first	$\mathbb{C}^{1\vert 2}$	$\mathfrak{sl}(1\vert 2)$	Z	Z_+	(0)
	second	$\mathbb{C}^{2\vert 0} \otimes (\mathbb{C}^{0\vert 2})^*$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	Z	(0)	Z_+
	second	$\mathbb{C}^{1\vert 1} \otimes (\mathbb{C}^{1\vert 1})^*$	$\mathfrak{sl}(1\vert 1) \oplus \mathfrak{sl}(1\vert 1)$	$\mathfrak{sl}(2)$	(0)	(0)

TABLE 5.

Proof. At once from Proposition 3.11 and Table 4. □

3.5.3. *Strange superalgebras.*

The gradings of $\mathfrak{s} = \mathfrak{psq}(n)$ are in a one-to-one correspondence with those of $\mathfrak{sl}(n) \simeq \mathfrak{s}_{\bar{0}} \simeq \mathfrak{s}_{\bar{1}}$ and are of depth one if one and only one root of the Dynkin diagram of $\mathfrak{sl}(n)$ is crossed.

$\mathfrak{s}$	Dynkin diagram of $\mathfrak{sl}(n)$	crossed root	$\mathfrak{s}^{-1} = (\mathfrak{s}^1)^*$	$\mathfrak{s}^0$
$\mathfrak{psq}(n)$ $n \geq 3$	$\overset{1}{\circ} - \cdots - \overset{1}{\circ}$	p-th	$\mathbb{C}^{p\|p} \circledcirc (\mathbb{C}^{n-p\|n-p})^*$	$\mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$

Table 6. The depth one gradings of $\mathfrak{psq}(n)$.

In this case $\mathrm{der}(\mathfrak{s}) = \mathfrak{pq}(n)$ and $\mathrm{out}(\mathfrak{s})$ is generated by an odd derivation D satisfying

$$D(\mathfrak{s}_{\bar 0}) = (0)\ , \qquad D\big|_{\mathfrak{s}_{\bar 1}} : \mathfrak{s}_{\bar 1} \mapsto \mathfrak{s}_{\bar 0} \ \text{ is an isomorphism of } \ \mathfrak{s}_{\bar 0} - \text{modules}\,.$$

It has degree zero for all gradings of $\mathfrak{s}$ and $\mathrm{der}^0(\mathfrak{s}) \simeq \mathfrak{p}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$.

The gradings of $\mathfrak{s} = \mathfrak{spe}(n)$ are in a one-to-one correspondence with the gradings of $\mathfrak{sl}(n) \simeq \mathfrak{spe}(n)_{\bar 0}$ and an integer k which determines the degree $\deg(F) = k$ of the highest weight vector F of the $\mathfrak{sl}(n)$-module $S^2(\mathbb{C}^n)$.

Table 7 below displays all depth one gradings of $\mathfrak{spe}(n)$; therein $\begin{pmatrix} n & 0 \\ 0 & n-2 \end{pmatrix}$ is the diagonal matrix of order $2(n-1)$ with the eigenvalues n and $n-2$, each of the same multiplicity $n-1$.

$\mathfrak{s}$	Dynkin diagram of $\mathfrak{sl}(n)$	crossed root	k	$\mathfrak{s}^{-1}$	$\mathfrak{s}^0$	$\mathfrak{s}^1$	$\mathfrak{s}^2$
$\mathfrak{spe}(n)$ $n \geq 3$	$\overset{1}{\circ} - \cdots - \overset{1}{\circ}$	none	-1	$\Pi(S^2(\mathbb{C}^n))$	$\mathfrak{sl}(n)$	$\Pi(\Lambda^2((\mathbb{C}^n)^*))$	(0)
		p-th	1	$\Pi(S^2(\mathbb{C}^{n-p\|p}))$	$\mathfrak{sl}(n-p\|p)$	$\Pi(\Lambda^2((\mathbb{C}^{n-p\|p})^*))$	(0)
		none	1	$\Pi(S^2(\mathbb{C}^{0\|n}))$	$\mathfrak{sl}(n)$	$\Pi(\Lambda^2((\mathbb{C}^{0\|n})^*))$	(0)
		first	2	$\mathbb{C}^{n-1\|n-1}$	$\mathfrak{spe}(n-1) \ni \begin{pmatrix} n & 0 \\ 0 & n-2 \end{pmatrix}$	$(\mathbb{C}^{n-1\|n-1})^*$	$\mathbb{C}^{0\|1}$

Table 7. The depth one gradings of $\mathfrak{spe}(n)$.

In this case $\mathrm{der}(\mathfrak{s}) \simeq \mathfrak{pe}(n)$ where $\mathrm{out}(\mathfrak{s}) = \mathfrak{t}_o$ is generated by the even derivation

$$D(\mathfrak{s}_{\bar 0}) = (0)\ , \qquad D\big|_{S^2(\mathbb{C}^n)} = \mathrm{Id}\ , \qquad D\big|_{\Lambda^2((\mathbb{C}^n)^*)} = -\,\mathrm{Id}\ .$$

It has always degree zero and $\mathrm{der}^0(\mathfrak{s}) \simeq \mathfrak{gl}(n-p|p)$ $(0 \leq p \leq n)$ or $\mathfrak{cpe}(n-1)$.

3.5.4. *Cartan superalgebras.*

The gradings of $W(n)$, $S(n)$, $\widetilde{S}(n)$ and $H(n)$ are all generated by gradings of type $\vec{k} = (k_1, \ldots, k_m)$ which satisfy appropriate restrictions on the spectrum (see [16] and also [2, §4.1]). The list of those of depth one is contained in e.g. [5, Proposition 9.1] and displayed in Table 8, with the action of $\mathfrak{s}^0$ on $\mathfrak{s}^{-1}$.

$\mathfrak{s}$	$\dim U^{-1}$	$\dim U^{0}$	$\dim U^{1}$	$\mathfrak{s}^{-1}$	$\mathfrak{s}^{0}$
$W(n)$ $n \geq 3$	1	$n-1$	0	$\Pi(W(n-1))$	$W(n-1) \inplus \Lambda(n-1)$
	0	r $(0 \leq r \leq n-1)$	$n-r$	$\Lambda(r) \otimes \mathbb{C}^{0\vert n-r}$	$W(r) \inplus \Lambda(r) \otimes \mathfrak{gl}(n-r)$
$S(n)$ $n \geq 4$	1	$n-1$	0	$\Pi(S(n-1))$	$S(W(n-1) \inplus \Lambda(n-1))$
	0	r $(0 \leq r \leq n-1)$	$n-r$	$\Lambda(r) \otimes \mathbb{C}^{0\vert n-r}$	$S(W(r) \inplus \Lambda(r) \otimes \mathfrak{gl}(n-r))$
$H(n)$ $n \geq 5$	0	0	n	$\mathbb{C}^{0\vert n}$	$\mathfrak{so}(n)$
	1	$n-2$	1	$\Pi(\Lambda(n-2))$	$H'(n-2) \inplus \Lambda(n-2)/\,\mathrm{vol}$
	0	$\frac{n}{2}$ (n even)	$\frac{n}{2}$ (n even)	$\Lambda(\frac{n}{2})/\mathbb{C}$	$W(\frac{n}{2})$

TABLE 8. The depth one gradings of the Cartan Lie superalgebras and their nonpositive parts.

Now $\mathfrak{s} = \mathrm{der}(\mathfrak{s})$ if $\mathfrak{s} = W(n)$. If $\mathfrak{s} = S(n)$ and $H(n)$ then $\mathrm{der}(\mathfrak{s})$ is isomorphic to, respectively, the superalgebra $CS(n)$ of vector fields of *constant* divergence and the *full* Hamiltonian superalgebra $H'(n)$ (the simple Lie superalgebra $H(n)$ is the derived ideal of $H'(n)$).

If $\mathfrak{s} = S(n)$ then $\mathrm{out}(\mathfrak{s})$ consists of the "principal" Euler supervector field

$$\sum_{\alpha=1}^{n} \xi^{\alpha} \frac{\partial}{\partial \xi^{\alpha}} \; ; \tag{3.10}$$

it has always degree zero and $\mathrm{der}^0(\mathfrak{s}) \simeq C(\mathfrak{s}^0)$.

If $\mathfrak{s} = H(n)$ then $\mathrm{out}(\mathfrak{s})$ is the two-dimensional solvable Lie superalgebra generated by (3.10) and the Hamiltonian supervector field

$$H_f = -(-1)^{p(f)} \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \xi^{\alpha}} \frac{\partial}{\partial \xi^{n+1-\alpha}} \tag{3.11}$$

with $f = \xi^1 \cdots \xi^n$. This supervector field has degree $n-2$ in the first grading of $H(n)$ in Table 8 and $\mathrm{der}^0(\mathfrak{s}) = \mathfrak{co}(n)$ in this case (only the Euler field contributes to the zero-degree part).

It has degree zero and, respectively, $n/2 - 1$ (n even) in the second and last gradings of $H(n)$ in Table 8 and contributes to $\mathrm{der}^0(\mathfrak{s})$ only in the second grading. A direct computation using the presentation $H'(n) \simeq \Lambda^+$ given by (3.11) and the Poisson bracket (see [8] for its definition and basic properties) says that $\mathrm{der}^0(\mathfrak{s}) = (H'(n-2) \Subset \Lambda(n-2)) \oplus G$ and $W(\frac{n}{2}) \oplus D$ in these two cases, where G is the "principal" grading operator acting on $\Lambda(n-2)$ and D is the grading operator of the last grading of $\mathfrak{s} = H(n)$ in Table 8.

The following is a direct consequence of the classification of the gradings of $\mathrm{der}(\mathfrak{s})$ carried out in §3.5.2-§3.5.4.

Proposition 3.13. *Let $\mathfrak{s}$ be a simple Lie superalgebra different from $\mathfrak{psl}(2|2)$. If $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ is a grading of depth one and $\mathrm{der}(\mathfrak{s}) = \bigoplus \mathrm{der}^p(\mathfrak{s})$ the associated grading of $\mathrm{der}(\mathfrak{s})$ then $d(\mathrm{der}(\mathfrak{s})) = 1$ and $\mathrm{der}^{-1}(\mathfrak{s}) = \mathfrak{s}^{-1}$. If $\mathfrak{s} = \mathfrak{psl}(2|2)$ then $d(\mathrm{der}(\mathfrak{s})) \leq 2$.*

The results of Proposition 3.11 and Proposition 3.13 together with Proposition 3.7 applied to the case where the socle is a simple Lie superalgebra $\mathfrak{s}$ immediately yield the following main result.

Theorem 3.14. *Any transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebra of depth 1 and with a simple socle $\mathfrak{s}$ is of the form $\mathfrak{g} = \mathfrak{s} \ni F$ where $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ is a grading of depth 1 of a simple Lie superalgebra and $F = \bigoplus F^p$ a graded subalgebra of $\mathrm{out}(\mathfrak{s})$ of depth $d(F) \leq 0$. Any grading of F satisfies $d(F) \leq 0$ (i.e. F is graded in nonnegative degrees) for all $\mathfrak{s} \neq \mathfrak{psl}(2|2)$.*

The gradings of depth 1 of the simple Lie superalgebras with the associated gradings of the algebras of outer derivations are classified in §3.5.2-§3.5.4. Theorem 3.14 gives therefore a complete description of the *transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebras of depth 1 with a simple socle.*

4. The general case

We now describe all transitive gradings $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of depth one of a Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{g}^1 \neq (0)$ and the representation $\mathrm{ad}_{\mathfrak{g}^0}|_{\mathfrak{g}^{-1}}$ is irreducible.

By Theorem 2.5, Proposition 3.2 and Proposition 3.7 $\mathfrak{g}$ is an intermediate admissible Lie superalgebra and the grading is induced by a grading of $\mathfrak{s}_{\max}$ generated by a grading of $\mathfrak{s}$ and a grading of Λ and W of type $\vec{k} = (k_1, \dots, k_m)$.

We start with the following preliminary result.

Proposition 4.1. *Any transitive and irreducible grading of depth 1 of an intermediate admissible Lie superalgebra $\mathfrak{g}$ with socle $\mathfrak{s} \otimes \Lambda$ is induced by a grading of $\mathfrak{s}_{\max}$ generated by:*

(I) *a depth one grading of $\mathfrak{s}$ and the trivial grading $\vec{k} = (0)$ of Λ and W or*

(II) *the trivial grading of $\mathfrak{s}$ and the $\vec{k} = (-1, 0)$ grading of Λ and W with $\dim U^{-1} = 1$.*

Proof. The inclusion $\mathfrak{g} \supset \mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$ immediately implies

$$\mathfrak{g}^p \supset (\mathfrak{s}^\Lambda)^p = \bigoplus_{k \in \mathbb{Z}} \mathfrak{s}^{p+k} \otimes \Lambda^{-k} ,$$

where $\Lambda = \bigoplus \Lambda^p$ and $W = \bigoplus W^p$ are the gradings of type $\vec{k} = (k_1, \dots, k_m)$ and $\mathfrak{s} = \bigoplus \mathfrak{s}^p$ is the grading of $\mathfrak{s}$. Since $d(\mathfrak{g}) = 1$ and Λ has always the identity element 1 with $\deg 1 = 0$, one gets two cases:

(I) $d(\mathfrak{s}) = 1$ and $d(\Lambda) = 0$,

(II) $d(\mathfrak{s}) \leq 0$.

We consider them separately.

Case (I). The subalgebra

$$\mathfrak{s}^0_{\max} = \bigoplus_{p<0} (\mathrm{der}^p(\mathfrak{s}) \otimes \Lambda^{-p}) \oplus (\mathrm{der}^0(\mathfrak{s}) \otimes \Lambda^0) \oplus W^0$$

stabilizes the nonzero subspace $\mathfrak{s}^{-1} \otimes \Lambda^0 \subset \mathfrak{s}^{-1}_{\max}$. In particular this subspace is $\mathfrak{g}^0$-stable and hence $\mathfrak{g}^{-1} = \mathfrak{s}^{-1} \otimes \Lambda^0$, by irreducibility of the grading of $\mathfrak{g}$. By $d(\Lambda) = 0$ and Lemma 3.6 one also has all $k_\alpha \geq 0$.

If $\mathfrak{g} = \mathfrak{s}_{\max} = \mathrm{der}(\mathfrak{s}) \otimes \Lambda \ni \mathbf{1} \otimes W$ one has $d(W) = 0$ by irreducibility and $\vec{k} = (0)$ by Lemma 3.6; the general case is an appropriate modification of this argument, as we will now see.

First of all assume $k_m > 0$ by contradiction. We recall that $\mathfrak{g}$ is admissible and that any constant supervector field $\partial_\xi \in \partial_V$ is related to an $x \in \mathfrak{g}$ such that $\pi(x) \in W = \partial_V \oplus \Lambda^+ \partial_V$ projects on ∂_ξ, i.e.

$$\begin{aligned} x &\equiv \pi(x) \mod \mathrm{der}(\mathfrak{s}) \otimes \Lambda , \\ \text{with } \pi(x) &\equiv \partial_\xi \mod \Lambda^+ \partial_V , \end{aligned} \tag{4.12}$$

where $\pi : \mathfrak{s}_{\max} \to W$ is the natural projection. Let $x \in \mathfrak{g}$ be related to ∂_{ξ^m}; one may assume $\deg(x) = -k_m$ since $\deg(\partial_{\xi^m}) = -k_m$ and $\mathfrak{g}$ is $\mathbb{Z}$-graded. This implies $x \in \mathfrak{g}^{-1}$ since $d(\mathfrak{g}) = 1$; however $x \notin \mathfrak{s}^{-1} \otimes \Lambda^0$, a contradiction.

It follows $k_m = 0$, $\vec{k} = (0)$ and $\mathfrak{g}^{-1} = \mathfrak{s}^{-1} \otimes \Lambda$.

Case (II). In this case $\mathfrak{s}_+ = \bigoplus_{p>0} \mathfrak{s}^p$ is a nilpotent ideal of $\mathfrak{s}$, hence $\mathfrak{s}_+ = (0)$ by simplicity of $\mathfrak{s}$ and both $\mathfrak{s} = \mathfrak{s}^0$ and $\operatorname{der}(\mathfrak{s}) = \operatorname{der}^0(\mathfrak{s})$ are trivially graded. If $d(\Lambda) \leq 0$ then $\mathfrak{s} \otimes \Lambda$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0 = \bigoplus_{p \geq 0} \mathfrak{g}^p$, a possibility which is not allowed by the transitivity of the grading of $\mathfrak{g}$. It follows that $d(\Lambda) = 1$ and $\vec{k} = (-1, k_2, \ldots, k_m)$ with $\dim U^{-1} = 1$, from Lemma 3.6.

Now, the subalgebra

$$\mathfrak{s}^0_{\max} = (\operatorname{der}(\mathfrak{s}) \otimes \Lambda^0) \oplus W^0$$

stabilizes the nontrivial subspace $\mathfrak{s} \otimes \Lambda^{-1} \subset \mathfrak{s}^{-1}_{\max}$ and therefore $\mathfrak{g}^{-1} = \mathfrak{s} \otimes \Lambda^{-1}$. This fact, admissibility of $\mathfrak{g}$ and a similar argument to the last part of (I) finally imply $\vec{k} = (-1, 0)$ with $\dim U^{-1} = 1$. □

It is convenient to have a closer look to the gradings of type (I) and (II) in the case $\mathfrak{g} = \mathfrak{s}_{\max}$. We recall that $\operatorname{out}(\mathfrak{s})$ is a graded subalgebra, by Proposition 3.9.

Case (I). Here $\mathfrak{s}^{-p}_{\max} = (0)$ for all $p \geq 3$ and

$$\begin{aligned}
\mathfrak{s}^{-2}_{\max} &= \operatorname{out}^{-2}(\mathfrak{s}) \otimes \Lambda, \\
\mathfrak{s}^{-1}_{\max} &= (\mathfrak{s}^{-1} \otimes \Lambda) \oplus (\operatorname{out}^{-1}(\mathfrak{s}) \otimes \Lambda), \\
\mathfrak{s}^{0}_{\max} &= (\mathfrak{s}^{0} \otimes \Lambda) \oplus (\operatorname{out}^{0}(\mathfrak{s}) \otimes \Lambda) \oplus W, \\
\mathfrak{s}^{1}_{\max} &= (\mathfrak{s}^{1} \otimes \Lambda) \oplus (\operatorname{out}^{1}(\mathfrak{s}) \otimes \Lambda), \\
\mathfrak{s}^{p}_{\max} &= (\mathfrak{s}^{p} \otimes \Lambda) \oplus (\operatorname{out}^{p}(\mathfrak{s}) \otimes \Lambda) \quad (p \geq 2),
\end{aligned} \tag{4.13}$$

where $\operatorname{out}^{-2}(\mathfrak{s}) = \operatorname{out}^{-1}(\mathfrak{s}) = (0)$ for all $\mathfrak{s}$ with the exception of $\mathfrak{s} = \mathfrak{psl}(2|2)$, see Proposition 3.13.

Case (II). The grading of Λ and W has type $\vec{k} = (-1, 0)$ and defined by the grading $U = U^{-1} \oplus U^0 = \mathbb{R}\xi \oplus E$. Hence $\Lambda = \Lambda^{-1} \oplus \Lambda^0$, $W = W^{-1} \oplus W^0 \oplus W^1$ where

$$\begin{aligned}
&\Lambda^{-1} = \xi\Lambda(E)\,, \qquad \Lambda^0 = \Lambda(E) \qquad \text{and} \\
&\qquad W^{-1} = \xi W(E)\,, \qquad W^0 = W(E) \oplus (\Lambda(E)\xi\partial_\xi)\,, \qquad W^1 = \Lambda(E)\partial_\xi\,.
\end{aligned}$$

Moreover $\mathfrak{s}^p_{\max} = (0)$ if $|p| \geq 2$ and

$$\begin{aligned}
\mathfrak{s}^{-1}_{\max} &= (\mathfrak{s} \otimes \xi\Lambda(E)) \oplus (\operatorname{out}(\mathfrak{s}) \otimes \xi\Lambda(E)) \oplus (\xi W(E)), \\
\mathfrak{s}^{0}_{\max} &= (\mathfrak{s} \otimes \Lambda(E)) \oplus (\operatorname{out}(\mathfrak{s}) \otimes \Lambda(E)) \oplus W(E) \oplus (\Lambda(E)\xi\partial_\xi), \\
\mathfrak{s}^{1}_{\max} &= \Lambda(E)\partial_\xi.
\end{aligned} \tag{4.14}$$

To state the main result Theorem 4.2 of this section we define

$$\mathrm{out}(\mathfrak{s}^\Lambda) := \mathrm{out}(\mathfrak{s}) \otimes \Lambda(n) \ni \mathbf{1} \otimes W(n)$$

and note that any subalgebra $\mathfrak{g}$ of the form

$$\begin{aligned}\mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda(n) \subset \mathfrak{g} \subset \mathfrak{s}_{\max} &= \mathrm{der}(\mathfrak{s}) \otimes \Lambda(n) \ni \mathbf{1} \otimes W(n) \\ &= \mathfrak{s}^\Lambda \ni \mathrm{out}(\mathfrak{s}^\Lambda) \;, \end{aligned} \tag{4.15}$$

can be written as $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ for a subalgebra F of $\mathrm{out}(\mathfrak{s}^\Lambda)$. The algebra $\mathfrak{g}$ is admissible if and only if F is admissible, that is the subalgebra $\pi(F)$ of W is admissible.

Theorem 4.2. *Let $\mathfrak{g} = \bigoplus_{p=-1}^{\ell} \mathfrak{g}^p$ be a transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebra of depth 1. Then $\mathfrak{g}$ is a semisimple Lie superalgebra with the socle $\mathrm{soc}(\mathfrak{g})$ given by $\mathrm{soc}(\mathfrak{g}) = \mathfrak{s}^\Lambda = \mathfrak{s} \otimes \Lambda$ where $\mathfrak{s}$ is a uniquely determined simple Lie superalgebra and $\Lambda = \Lambda(n)$ is the Grassmann algebra for some nonnegative integer n. The superalgebra $\mathfrak{g}$ is a graded subalgebra of the Lie superalgebra*

$$\mathrm{der}(\mathfrak{s}^\Lambda) = \mathfrak{s}^\Lambda \ni \mathrm{out}(\mathfrak{s}^\Lambda) \;,$$

where $\mathrm{out}(\mathfrak{s}^\Lambda) = \mathrm{out}(\mathfrak{s}) \otimes \Lambda \ni 1 \otimes W$, with one of the following gradings:

- (I) *$\mathfrak{s} = \bigoplus \mathfrak{s}^p$ has a $\mathbb{Z}$-grading of depth 1 and $\Lambda = \Lambda^0$ and $W = W^0$ have the trivial gradings of type $\vec{k} = (0)$;*
- (II) *$\mathfrak{s} = \mathfrak{s}^0$ has the trivial grading and Λ and W have the gradings of type $\vec{k} = (-1, 0)$ with $\dim U^{-1} = 1$.*

Moreover $\mathfrak{g}$ can be written as a semidirect sum $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$, where $F = \bigoplus F^p$ is a nonnegatively graded subalgebra of the Lie superalgebra $\mathrm{out}(s^\Lambda)$ which is admissible, that is the natural projection from F to ∂_V is surjective.

Conversely any grading as in (I) or (II) defines a transitive nonlinear and irreducible grading of depth 1 of the Lie superalgebra $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ for any nonnegatively graded subalgebra F of $\mathrm{out}(\mathfrak{s}^\Lambda)$ which is admissible.

Proof. By Theorem 2.5 and Proposition 3.2 any transitive nonlinear irreducible graded Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of depth 1 is an admissible intermediate Lie superalgebra (4.15) and therefore $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ for an admissible graded subalgebra F of $\mathrm{out}(\mathfrak{s}^\Lambda)$. By Proposition 4.1 and its proof the grading of $\mathfrak{g}$ is induced by a grading of $\mathfrak{s}_{\max}$ either generated by the grading (I) with $\mathfrak{g}^{-1} = \mathfrak{s}^{-1} \otimes \Lambda$ or by the grading (II) with $\mathfrak{g}^{-1} = \mathfrak{s} \otimes \xi\Lambda(E)$. This fact and equations (4.13)-(4.14) immediately imply that F is graded in nonnegative degrees.

Conversely let $\mathfrak{s}_{\max}$ be graded as in (I) or (II) and $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ the graded Lie superalgebra determined by an admissible subalgebra F of $\mathrm{out}(\mathfrak{s}^\Lambda)$ which is graded in nonnegative degrees. It is clear that $d(\mathfrak{g}) = 1$; we now show that the grading of $\mathfrak{g}$ is also transitive, irreducible and nonlinear.

(*Nonlinearity*) In case (I) $\mathfrak{s}^1 \neq (0)$ from Lemma 3.10 and $\mathfrak{g}^1 \supset \mathfrak{s}^1 \otimes \Lambda \supset \mathfrak{s}^1$. In case (II) there exists by admissibility a vector $x \in \mathfrak{g}$ which projects to the constant supervector field ∂_ξ. Since $\mathfrak{g}$ is $\mathbb{Z}$-graded one may assume $\deg(x) = \deg(\partial_\xi) = 1$.

(*Transitivity*) In case (I) transitivity is a consequence of the inclusions

$$\mathfrak{g}^{-1} = \mathfrak{s}^{-1} \otimes \Lambda \ , \qquad \mathfrak{g}^0 \subset (\mathrm{der}^0(\mathfrak{s}) \otimes \Lambda) \oplus W \ , \qquad \mathfrak{g}^p \subset \mathrm{der}^p(\mathfrak{s}) \otimes \Lambda \ \ (p \geq 1)$$

and part (i) of Lemma 3.10. In case (II) the adjoint action of

$$\begin{aligned} \mathfrak{s}^0_{\max} &= (\mathrm{der}(\mathfrak{s}) \otimes \Lambda(E)) \oplus (\Lambda(E)\xi\partial_\xi) \oplus W(E) \\ &\simeq ((\mathrm{der}(\mathfrak{s}) \oplus \xi\partial_\xi) \otimes \Lambda(E)) \oplus W(E) \end{aligned}$$

on $\mathfrak{g}^{-1} \simeq (\mathfrak{s}\xi)\otimes\Lambda(E)$ is given by the action on $\mathfrak{s}\xi$ of the "conformal extension" $\mathrm{der}(\mathfrak{s})\oplus\xi\partial_\xi$ of $\mathrm{der}(\mathfrak{s})$ and the natural left actions on $\Lambda(E)$ of $\Lambda(E)$ and $W(E)$; this easily implies transitivity for $\mathfrak{g}^0 \subset \mathfrak{s}^0_{\max}$. Transitivity for $\mathfrak{g}^1$ is immediate.

(*Irreducibility*) We start with (I) and a nonzero submodule $\mathfrak{m} \subset \mathfrak{s}^{-1}\otimes\Lambda$ for $\mathfrak{g}^0$. By (v) of Lemma 3.10, it is always possibile to find $x \in \mathfrak{m}$ and $y \in \mathfrak{s}^0\otimes\Lambda \subset \mathfrak{g}^0$ such that $[x, y] \in \mathfrak{m}$ is nonzero and of top degree, i.e. $x \in \mathfrak{s}^{-1}\otimes\Lambda^{(n)}$ where here $\Lambda = \bigoplus \Lambda^{(p)}$ denotes the usual principal grading of the Grassmann algebra Λ.

On the other hand $\mathfrak{g}^{-1}$ is isomorphic to the sum of 2^n copies of $\mathfrak{s}^{-1}$ as an $\mathfrak{s}^0$-module and if $\mathfrak{m}$ contains a nonzero element of a copy then it includes the full copy too, by (iii) of Lemma 3.10. Hence $\mathfrak{m} \supset \mathfrak{s}^{-1} \otimes \Lambda^{(n)}$.

By admissibility and $\deg(W) = 0$ any constant supervector field is related to an element of $\mathfrak{g}^0$; we denote by $\mathcal{F} \subset \mathfrak{g}^0$ the collection of all these elements. Since $\mathfrak{m} \supset \mathfrak{s}^{-1} \otimes \Lambda^{(n)}$ and every $x \in \mathcal{F}$ is of the form (4.12), one gets that $\mathfrak{m}$ contains a nonzero element also in every copy $\mathfrak{s}^{-1}$ of $\mathfrak{s}^{-1} \otimes \Lambda^{(n-1)}$ and

$$\mathfrak{m} \supset \bigoplus_{p\geq n-1} \mathfrak{s}^{-1} \otimes \Lambda^{(p)} \ .$$

A repeated application of this argument yields $\mathfrak{m} \supset \mathfrak{s}^{-1}\otimes\Lambda = \mathfrak{g}^{-1}$ and proves irreducibility.

The proof is similar in case (II) where $\mathcal{F}$ is now replaced by the collection of all the elements of $\mathfrak{g}^0$ which are related to a constant supervector field in $W(E)$. We omit the details. □

Remark 4.3. The proof of the irreducibility in Theorem 4.2 works also for those $\mathfrak{g}^0$-submodules of $\mathfrak{g}^{-1}$ which are not necessarily $\mathbb{Z}_2$-graded. Therefore the representation $\mathrm{ad}_{\mathfrak{g}^0}|_{\mathfrak{g}^{-1}}$ is always an irreducible representation of G-type.

Theorem 4.2 reduces the description of the transitive nonlinear irreducible $\mathbb{Z}$-graded Lie superalgebras $\mathfrak{g}$ of depth one to the description of the nonnegatively graded subalgebras $F = F^0\oplus F^1\oplus\cdots$ of $\mathrm{out}(\mathfrak{s}^\Lambda) = \mathrm{out}(\mathfrak{s})\otimes\Lambda \ni 1\otimes W$ with admissible subalgebra $\pi(F) \subset W$, where $\mathrm{der}(\mathfrak{s}^\Lambda)$ has the gradings (I) or (II). Let us assume $\mathfrak{s} \neq \mathfrak{psl}(2|2)$ for simplicity of exposition.

In the grading (I) there are the two extreme cases $\mathrm{out}(\mathfrak{s}) = (0)$ and $U = (0)$. If $\mathrm{out}(\mathfrak{s}) = (0)$ then $\mathfrak{g} = \mathfrak{s}^\Lambda \ni F$ for *any* admissible subalgebra F of W. Such subalgebras can be described as follows. Let

$$\varphi : \partial_V \longrightarrow W = \partial_V \oplus \Lambda^+ \partial_V \ , \qquad \varphi(\partial_{\xi^i}) = \partial_{\xi^i} + \sum_j f_i^j \partial_{\xi^j} \ ,$$

be a section of the natural projection from $W = \partial_V \oplus \Lambda^+ \partial_V$ onto ∂_V, where $f_i^j \in (\Lambda^+)_{\bar{0}}$ is an even nilpotent superfunction on the purely odd n-dimensional linear supermanifold with coordinates $\{\xi^i\}$, for all $i, j = 1, \ldots, n$. Then $F \subset W$ is any Lie superalgebra which contains $\varphi(\partial_V)$, in particular, the Lie subalgebra generated by $\varphi(\partial_V)$ in W.

If $U = (0)$ then $\mathfrak{g} \subset \mathrm{der}(\mathfrak{s})$ and $\mathfrak{g} = \mathfrak{s} \ni F$ for *any* $\mathbb{Z}$-graded subalgebra F of $\mathrm{out}(\mathfrak{s})$ (for this case see also Table 2, §3.5.2-§3.5.4 and Theorem 3.14).

In general there is an exact sequence of $\mathbb{Z}$-graded Lie superalgebras

$$0 \longrightarrow F \cap \mathrm{out}(\mathfrak{s}) \otimes \Lambda \longrightarrow F \longrightarrow F' \longrightarrow 0 \ , \tag{4.16}$$

where F' is an admissible subalgebra of W. The first term of (4.16) can also be in turn described through the exact sequence,

$$0 \longrightarrow F \cap \mathrm{out}(\mathfrak{s}) \otimes \Lambda^+ \longrightarrow F \cap \mathrm{out}(\mathfrak{s}) \otimes \Lambda \longrightarrow F'' \ , \tag{4.17}$$

where F'' is a $\mathbb{Z}$-graded subalgebra of $\mathrm{out}(\mathfrak{s})$.

In the grading (II) of Theorem 4.2 there is the extreme case $E = (0)$ where

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \qquad \text{and}$$

$$\mathfrak{g}^{-1} = \mathfrak{s}\xi \ , \qquad \mathfrak{g}^0 = \mathfrak{s} \ni \widetilde{F} \ , \qquad \mathfrak{g}^1 = \mathbb{C}\partial_\xi \ . \tag{4.18}$$

Here $\widetilde{F}$ is *any* subalgebra of the direct sum $\mathrm{out}(\mathfrak{s}) \oplus \mathbb{C}\xi\partial_\xi$ of $\mathrm{out}(\mathfrak{s})$ with the space generated by the grading operator $-\xi\partial_\xi$ of (4.18). We note that the Lie superalgebra (4.18) is a generalization of the Lie superalgebra "G^z" and, at the same time, the Lie superalgebra "H^ξ" introduced in [16, p. 71].

In general there is a graded ideal

$$\mathfrak{i} = (\mathrm{out}(\mathfrak{s}) \otimes \Lambda(E)) \oplus (\Lambda(E)\xi\partial_\xi) \oplus (\Lambda(E)\partial_\xi) \tag{4.19}$$

of $\mathrm{out}_0(\mathfrak{s}^\Lambda) = \bigoplus_{p \geq 0} \mathrm{out}^p(\mathfrak{s}^\Lambda)$ and an exact sequence

$$0 \longrightarrow F \cap \mathfrak{i} \longrightarrow F \longrightarrow F' \longrightarrow 0 \ , \tag{4.20}$$

where F' is an admissible subalgebra of $W(E)$. These subalgebras can be described similarly as above in terms of sections $\varphi' : \partial_E \longrightarrow W(E)$ of the natural projection of $W(E) = \partial_E \oplus \Lambda^+(E)\partial_E$ onto ∂_E. If $\mathfrak{i}^+$ is the ideal of $\mathfrak{i}$ obtained by replacing $\Lambda(E)$ with $\Lambda^+(E)$ in (4.19) then the first term of (4.20) fits into the exact sequence

$$0 \longrightarrow F \cap \mathfrak{i}^+ \longrightarrow F \cap \mathfrak{i} \longrightarrow F'' \ , \tag{4.21}$$

where $F'' = (F'')^0 \oplus \mathbb{C}\partial_\xi$ and $(F'')^0$ a is subalgebra of $\mathrm{out}(\mathfrak{s}) \oplus \mathbb{C}\xi\partial_\xi$.

The extension problems associated with these exact sequences look rather complicated and will not be addressed here.

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